Regularity and global existence for the nonlinear Schrödinger equation

In Theorem 12.5 we have constructed $H^1$–solutions $u \in C^1(J(u_0), H^{-1}(\mathbb{R}^d)) \cap C(J(u_0), H^1(\mathbb{R}^d))$ of the nonlinear Schrödinger equation

$$u'(t) = i\Delta u(t) - i\mu|u(t)|^{\alpha-1}u(t), \quad t \in J(u_0), \quad u(0) = u_0, \quad (13.1)$$

for $u_0 \in H^1(\mathbb{R}^d)$, the maximal existence interval $J(u_0)$, $\mu \in \{-1, 1\}$ and $1 < \alpha < \alpha_c = \frac{d+2}{d-2}$. As we saw in Example 10.3, there is a blow-up solution in the focusing case $\mu = -1$ for $\alpha = 1 + \frac{4}{d}$. In this lecture we present three positive results on global existence for (13.1), namely:

(a) Defocusing case: If $\mu = 1$, then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1(\mathbb{R}^d)$ and $\alpha \in (1, \alpha_c)$.

(b) Small $\alpha$: If $\alpha \in (1, 1 + \frac{4}{d})$, then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1(\mathbb{R}^d)$.

(c) Small $u_0$: There is a radius $\rho > 0$ such that $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1(\mathbb{R}^d)$ with $\|u_0\|_{1,2} \leq \rho$, where $\alpha \in (1, \alpha_c)$.

In all three cases the nonlinearity is relatively tame so that it does not destroy the global existence that we have in the linear case: In (a) we have the right sign $\mu = 1$. In (b) the nonlinearity does not grow too much. In (c) the solution $u$ is small initially which leads to an even smaller nonlinearity $|u|^{\alpha-1}u$. These results are shown in Theorem 13.3. Moreover, the proof of (c) leads to the Lyapunov stability of the solution $u_* = 0$ of (13.1), see Corollary 13.4.

The basic criteria (a) – (c) for global existence rely on the conservation laws for the energy and for the $L^2$–norm of $H^1$–solutions. In Remarks 10.4 and 10.5 we derived them only for $u \in C^1(J(u_0), H^1(\mathbb{R}^d)) \cap C(J(u_0), H^2(\mathbb{R}^d))$. We show the conservation laws for $H^1$–solutions in Theorem 13.2 by various approximation arguments. In this proof we work with $H^2$–solutions. For our reasoning it is crucial to know that the maximal $H^1$–solution of (13.1) is even an $H^2$–solution on its full existence interval $J(u_0)$ provided that $u_0 \in H^2(\mathbb{R}^d)$. Thus, the solution $u(t)$ preserves the initial regularity $u_0 \in H^2(\mathbb{R}^d)$ for all $t \in J(u_0)$, which is of independent interest.

To prove this fact, as in Theorem 12.5 we look for a fixed point $v$ satisfying

$$v = T(\cdot)u_0 + T_+ F(v) = \Phi_{u_0}(v),$$

where $F(v) = -i\mu|v|^{\alpha-1}v$ and $T(\cdot)$ is the free Schrödinger group. In contrast to Lemma 12.4 we now consider $\Phi$ on a space involving differentiability of $v$ in time. The resulting fixed point then turns out to be an $H^2$–solution and it coincides with the known $H^1$–solution $u$ from Theorem 12.5.
At the end of the lecture we briefly discuss further results on blow-up and global existence.

We now come to the announced regularity theorem. As in the previous lectures, we set \( p = 1 + \alpha < \frac{d+2}{(d-2)} \), \( \mu \in \{-1, 1\} \), and \( u_0 \in H^2(\mathbb{R}^d) \). Then the maximal \( H^1 \)-solution of (13.1) is even an \( H^2 \)-solution on \( J(u_0) \), i.e. \( u \in C^1(J(u_0), L^2(\mathbb{R}^d)) \cap C(J(u_0), H^2(\mathbb{R}^d)) \). It further satisfies \( u \in W_{q,k}^1(I, L^p(\mathbb{R}^d)) \) for all open bounded intervals \( I \) with \( \overline{I} \subseteq J(u_0) \).

**Proof.** Let \( u_0 \in H^2(\mathbb{R}^d) \) and let \( u \) be the maximal \( H^1 \)-solution of (13.1) obtained in Theorem 12.5. Take any compact interval \( J_0 \subseteq J(u_0) \) containing 0. We have to show that \( u \) is an \( H^2 \)-solution on \( J_0 \) with \( u \in W_{q,k}^1(J_0, L^p(\mathbb{R}^d)) \).

By a refinement of the fixed point argument in the proof of Theorem 12.5, we first prove this claim on an interval \( J_1 = [-b_1, b_1] \). It turns out that this time \( b_1 > 0 \) essentially depends on

\[
\overline{p} = \max_{t \in J_0} \|u(t)\|_{1,2}
\]

and in particular not on \( \|u_0\|_{2,2} \). We can thus repeat the argument for the initial values \( u(\pm b_1) \) with the same time step size \( b_1 \) and deduce the assertion in finitely many iterations. Throughout we use the setting and the notation of the proof of Theorem 12.5.

1) We fix \( r = 1 + C_{S_0, \overline{p}} \). Let \( b > 0 \). We set \( J = (-b, b) \) and use the spaces

\[
E_k(b) = L^\infty(J, H^k(\mathbb{R}^d)) \cap L^2(J, W_0^k(\mathbb{R}^d)),
\]

endowed with the norms given by

\[
\|v\|_{k,b} = \max\{\|v\|_{L^\infty(J, H^k)}, \|v\|_{L^2(J, W_0^k)}\},
\]

where \( k \in \{0, 1\} \). Lemma 12.4 says that the \( H^1 \)-solution \( u \) is a fixed point of the operator \( \Phi \) (see (13.3)) in the set \( \Sigma(r, b) = \overline{B_{E_k(b)}(0, r)} \), where \( 0 < b \leq b_0(\|u_0\|_{1,2}) \). We will show that it is also a fixed point in a subset of more regular functions, using that \( u_0 \in H^2(\mathbb{R}^d) \). To this aim, for \( R > 0 \) (to be fixed below) we define the spaces

\[
\mathcal{F}(b) = E_k(b) \cap W_{q,k}^1(J, L^p(\mathbb{R}^d)) \cap W_{\infty,k}^1(J, L^2(\mathbb{R}^d)),
\]

\[
\Theta = \Theta(R, b) = \{v \in \mathcal{F}(b) \mid v(0) = u_0, \|v\|_{1,b} \leq r, \|v'\|_{0,b} \leq R\}.
\]

Since \( u_0 \in H^2(\mathbb{R}^d) \) and \( H^2(\mathbb{R}^d) \hookrightarrow W_{p,k}^1(\mathbb{R}^d) \) by Sobolev’s embedding (5.5), there is a number \( \beta(\overline{p}) \) such that the function \( v \) given by \( v(t) = u_0 \) for \( |t| < b \) belongs to \( \Theta(b, R) \) for all \( b \in (0, \beta(\overline{p})) \) and \( R > 0 \). Hence, \( \Theta(b, R) \neq \emptyset \) in this case.

We endow \( \Theta \) with the metric given by \( \|v - w\|_{0,b} \). We recall from Remark 12.6 (d) that \( W_{\infty,k}^1(J, L^2(\mathbb{R}^d)) \) is isomorphic to the space of Lipschitz functions \( f : J \to L^2(\mathbb{R}^d) \). Moreover, for \( v \) in these isomorphic spaces the Lipschitz constant of \( v \) coincides with \( \|v\|_{L^\infty(J, L^2)} \). We first claim that \( \Theta \) is complete.
In fact, take a Cauchy sequence \((v_n)\) in \(\Theta\). In Lemma 12.2 we have seen that \((v_n)\) converges in \(\mathcal{E}_0(b)\) to a function \(v \in \mathcal{E}_1(b)\) with \(\|v\|_{1,b} \leq r\) as \(n \to \infty\). Since the maps \(v_n : J \to L^2(\mathbb{R}^d)\) converge in \(L^\infty(J,L^2(\mathbb{R}^d))\) to \(v\) and are uniformly Lipschitz with bound \(R\), we conclude that \((v_n)\) tends to \(v\) in \(C(J,L^2(\mathbb{R}^d))\) as \(n \to \infty\), that \(v(0) = u_0\) and that \(v : J \to L^2(\mathbb{R}^d)\) is Lipschitz with bound \(R\). Using again Remark 12.6 (d), we see that the function \(v\) belongs to \(W^1_{\infty}(J,L^2(\mathbb{R}^d))\) with \(\|v'\|_{L^\infty(J,L^2)} \leq R\). Further, after passing to a subsequence, \((v_n)\) tends weakly in \(W^1_q(J,L^p(\mathbb{R}^d))\) to a function \(w \in W^1_q(J,L^p(\mathbb{R}^d))\) as \(j \to \infty\) with \(\|w'\|_{L_q(J,L^p)} \leq R\). In particular, \(v_n\) and \(v'_n\) converge weakly in \(L^q(J,L^p(\mathbb{R}^d))\) to \(w\) and \(w'\), respectively. We thus obtain \(v = w\), \(v \in \mathcal{F}(b)\) and \(\|v'\|_{0,b} \leq R\).

Summing up, \(v \in \Theta\) and \(\Theta\) is complete.

2) Let \(v,w \in \Theta(R,b)\) for any \(b \in (0,b_0(p)]\), see (12.16). We define again
\[
\Phi(v)(t) = T(t)u_0 + \int_0^t T(t-s)F(v(s)) \, ds = T(t)u_0 + \int_0^t T(s)F(v(t-s)) \, ds
\]
for \(t \in J = [-b,b]\). We fix below \(R\) and \(b\) so that \(\Phi : \Theta(R,b) \to \Theta(R,b)\) is a strict contraction. In the proof of Lemma 12.4 we have already shown that \(\Phi(v) \in \mathcal{E}_1(b) \cap C(J,H^1(\mathbb{R}^d))\),
\[
\|\Phi(v) - \Phi(w)\|_{0,b} \leq \frac{1}{2} \|v - w\|_{0,b} \quad \text{and} \quad \|\Phi(v)\|_{1,b} \leq r,
\]
see (12.14) – (12.16).

3) We next prove that \(\frac{d}{dt} \Phi(v) \in \mathcal{E}_0(b)\) with \(\|\frac{d}{dt} \Phi(v)\|_{0,b} \leq R\) for all \(v \in \Theta(R,b)\). To this aim, we first differentiate the integral in (13.2) with respect to \(t\). This is done via an approximation argument. Corollary 9.3 shows that \(F : L^p(\mathbb{R}^d) \to L^{q'}(\mathbb{R}^d)\) is (real) continuously differentiable with derivative given by \(F'(\varphi) \psi = \varphi'(\varphi) \psi\) for \(\varphi, \psi \in L^p(\mathbb{R}^d)\) and \(\varphi(z) = -i \mu |z|^\alpha - 1 z\) for \(z \in \mathbb{R}^2\). Moreover,
\[
\|F'(\varphi) \psi\|_{p'} \leq c_1 \|\varphi\|_p^{\alpha-1} \|\psi\|_p.
\]
Here and below \(c_j\) are constants only depending on \(\alpha\) and \(d\). Lemma 12.7 allows to approximate \(v\) in \(W^1_q(J,L^p(\mathbb{R}^d))\) by \(w_n \in C(J,L^p(\mathbb{R}^d))\). Passing to a subsequence if necessary, we may further assume that \(w'_n(t)\) converges in \(L^p(\mathbb{R}^d)\) as \(n \to \infty\) and \(\|w'_n(t)\|_p \leq h(t)\) for all \(n \in \mathbb{N}\), a.e. \(t \in J\) and a function \(h \in L^q(J) \hookrightarrow L^q(J)\), where we note that \(q' < 2 < q\). Finally, taking \(a = 0\) and \(J = (-b,b)\) in the proof of Lemma 12.7 (b), we see that \(w_n(0) = v(0) = u_0\).

Lemma 12.7 also yields that \((w_n)\) converges to \(v\) in \(C(J,L^p(\mathbb{R}^d))\). It is thus bounded by a constant \(\sigma\) in this space. The properties of \(F\) yield that the functions \(F'(w_n(t))w'_n(t)\) tend to \(F'(v(t))v'(t)\) in \(L^{q'}(\mathbb{R}^d)\) as \(n \to \infty\) for a.e. \(t \in J\) and that
\[
\sup_{n \in \mathbb{N}} \|F'(w_n(t))w'_n(t)\|_{p'} \leq \sup_{s \in J} c_1 \|w_n(s)\|_p^{\alpha-1} \|w'_n(t)\|_p \leq c_1 \sigma^{\alpha-1} h(t)
\]
for a.e. \(t \in J\). From dominated convergence we deduce that \(F'(w_n)w'_n \to F'(v)v'\) in \(L^{q'}(J,L^{q'}(\mathbb{R}^d))\) as \(n \to \infty\).
Since $L^p(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$, we have $F(w_n) \in C^1(\mathcal{J}, H^{-1}(\mathbb{R}^d))$ and so the derivative
\[
\frac{d}{dt} \int_0^t T(s)F(w_n(t-s)) \, ds = T(t)F(w_n(0)) + \int_0^t T(s)F'(w_n(t-s))w'_n(t-s) \, ds \nabla
\]
exists in $H^{-1}(\mathbb{R}^d)$. (In this calculation we identify $\mathbb{C}$ with $\mathbb{R}^2$.) Due to Strichartz’ estimate (12.6), the right-hand side of the above identity is continuous in $L^2(\mathbb{R}^d)$ and converges to
\[
T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))v'(s) \, ds
\]
in $L^2(\mathbb{R}^d)$ uniformly in $t$ as $n \to \infty$. Similarly, the integral on the left-hand side tends to $T_\tau^* F(v)(t)$ in $L^2(\mathbb{R}^d)$ uniformly in $t$. We can thus differentiate the integral in (13.2) in $L^2(\mathbb{R}^d)$ and obtain
\[
\frac{d}{dt} \int_0^t T(s)F(v(t-s)) \, ds = T(t)F(u_0) + \int_0^t T(t-s)F'(v(s))v'(s) \, ds,
\]
\[
\frac{d}{dt} \Phi(v)(t) = T(t)(i\Delta u_0 + F(u_0)) + \int_0^t T(t-s)F'(v(s))v'(s) \, ds \tag{13.5}
\]
for all $t \in J$.

4) In this step we establish that $\frac{d}{dt} \Phi(v) \in \mathcal{E}_0(b)$ and estimate its norm. It is crucial that $R$ will enter only linearly. Using inequality (13.4), Sobolev’s embedding (12.4) and $\|v(s)\|_{1,2} \leq r$, we derive
\[
\|F'(v(s))v'(s)\|_{p'} \leq c_1 C_{\alpha^{-1}}^\alpha r^\alpha - 1 \|v'(s)\|_p
\]
for all $s \in J$. Strichartz’ estimate (12.6) and Hölder’s inequality now allow to bound the $\mathcal{E}_0(b)$–norm of the integral term in (13.5) by
\[
C_{St} \|F'(v)v\|_{L^p(J,L^p')} \leq C_{St} c_1 C_{\alpha^{-1}}^\alpha r^\alpha - 1 \|v'(s)\|_{L^p(J,L^p')} \leq c_2 r^\alpha - 1 b^\frac{1}{2} - \frac{1}{q} \|v'(s)\|_{L^q(J,L^q)} \leq c_2 r^\alpha - 1 b^\frac{1}{2} - \frac{1}{q} R \tag{13.6}
\]
with $c_2 := 2^\frac{1}{2} - \frac{1}{2} c_1 C_{\alpha^{-1}}^\alpha$, using $v \in \Theta(R,b)$. We further recall that $H^2(\mathbb{R}^d) \hookrightarrow L^{2a}(\mathbb{R}^d)$ by Sobolev’s embedding (5.5) since $\alpha < \frac{d-1}{d}$. Hence, $\|F(u_0)\|_2 = \|u_0\|_{2\alpha} \leq c_3 \|u_0\|_{2,2}$. Strichartz’ estimates, (13.5) and (13.6) thus yield that
\[
\|\frac{d}{dt} \Phi(v)\|_{0,b} \leq C_{St}(\|u_0\|_{2,2} + c_3 \|u_0\|_{2,2}) + c_2 r^\alpha - 1 b^\frac{1}{2} - \frac{1}{q} R \tag{13.7}
\]
and that $\frac{d}{dt} \Phi(v)$ belongs to $C(\mathcal{J}, L^2(\mathbb{R}^d))$. We fix $R = 2C_{St}(\|u_0\|_{2,2} + c_3 \|u_0\|_{2,2})$ and choose
\[
b_1 = \min\{b_0(\mathcal{P}), b(\mathcal{P}), (2c_2 r^\alpha - 1)^\frac{q}{q-\alpha}\}.
\]
Since $r = 1 + C_{St} \rho$, the number $b_1$ only depends on $\rho$, $\alpha$ and $d$. The inequalities (13.3) and (13.7) now show that $\Phi : \Theta(R,b_1) \to \Theta(R,b_1)$ is a strict contraction. We thus obtain a fixed point $v_\tau = \Phi(v_\tau) \in \Theta(R,b_1) \subseteq F(b_1)$ with $v_\tau \in C(\mathcal{J}_1, H^1(\mathbb{R}^d)) \cap C^1(\mathcal{J}_1, L^2(\mathbb{R}^d)) \cap W_0^1(\mathcal{J}_1, L^q(\mathbb{R}^d))$, where $\mathcal{J}_1 := [-b_1, b_1]$. 


By Lemma 12.4, the $H^1$–solution $u$ is the only fixed point of $\Phi$ in $\Sigma(r,b_1) \supseteq \Theta(R,b_1)$. Hence, $u = u_0 \in C^1(J_1, L^2(\mathbb{R}^d)) \cap C(J_1, H^1(\mathbb{R}^d)) \cap W^1_q(J_1, L^p(\mathbb{R}^d))$.

5) We still have to show that $u \in C(J_1, H^2(\mathbb{R}^d))$. To prove this, we use a “boot-strapping” argument based on the following fact, see Theorems 4.3.8 (ii) and 4.3.10 (ii) in [Kry08].

Let $r, s \in (1, \infty)$. Then the operator $I - \Delta : W^s_2(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ is invertible with bounded inverse $R_r$. If $g \in L^r(\mathbb{R}^d) \cap L^s(\mathbb{R}^d)$, then $R_rg = R_g$. We thus write $(I - \Delta)^{-1}$ instead of $R_r$.

As a starting point we note that (13.1) yields

$$u - \Delta u = u + iu' - iF(u) = f + g,$$

where $f := u + u' \in C(J_1, L^2(\mathbb{R}^d))$ and $g := -iF(u) \in C(J_1, L^\infty(\mathbb{R}^d))$ since $u \in C^1(J_1, L^2(\mathbb{R}^d)) \cap C(J_1, L^p(\mathbb{R}^d))$ and $p' = \frac{p}{\alpha}$. Note that $\frac{p}{\alpha} \in (1, 2)$. The above stated regularity result for $\Delta$ shows that $(I - \Delta)^{-1} f \in C(J_1, H^2(\mathbb{R}^d))$ and $(I - \Delta)^{-1} g \in C(J_1, W^2_\alpha(\mathbb{R}^d))$. Sobolev’s embedding (5.5) thus yields

$$u = (I - \Delta)^{-1}(f + g) \in C(J_1, L^r(\mathbb{R}^d))$$

with $r_1 = \frac{d}{\alpha - 2p} p =: \gamma \alpha > 2 > 2p$ and any $r_1 \in (2, \infty)$ otherwise. Note that $\gamma > 1$ if $\alpha > 2p$ since $p = \alpha + 1$ and $\alpha < \frac{d+2}{(d-2)}$.

This extra integrability of $u$ implies that $g \in C(J_1, L^{d-\frac{d-1}{\alpha}}(\mathbb{R}^d))$. If $r_1 \geq 2\alpha$, we obtain that $g$ belongs to $C(J_1, L^2(\mathbb{R}^d))$ since then $L^\infty(\mathbb{R}^d) \cap L^{d-\frac{d-1}{\alpha}}(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$.

As a result, $u = (I - \Delta)^{-1}(f + g) \in C(J_1, H^2(\mathbb{R}^d))$.

If $r_1 < 2\alpha$, as above we infer that $u \in C(J_1, L^2(\mathbb{R}^d))$ with $r_2 = \frac{d}{\alpha - 2r_1} \geq \gamma r_1 = \gamma^2 \alpha > 2r_1$ and with any $r_2 \in (2, \infty)$ if $\alpha > 2r_1$. Since $\gamma > 1$, in finitely many steps we arrive at $r_1 \geq \gamma^m \alpha \geq 2\alpha$, and hence $u \in C(J_1, H^2(\mathbb{R}^d))$.

6) We can now finish the proof. If $J_0 \subseteq J_1$ we are done. If not, assume that max $J_0 > b_1$. Since $u(b_1) \in H^2(\mathbb{R}^d)$, we can repeat steps 2) – 5) with initial value $u(b_1)$ and the same time step $b_1$. We then obtain an $H^2$–solution $u_1$ of (13.1) on $[b_1, 2b_1]$ with $u_1(b_1) = u(b_1)$. By Lemma 12.3, we can glue together these functions to an $H^2$–solution $v$ on $[-b_1, 2b_1]$ with $v \in W^2_q(([-b_1, 2b_1], L^p(\mathbb{R}^d)))$.

The uniqueness of $H^1$–solutions yields that $v = u$ on $[-b_1, 2b_1]$. We can iterate this procedure and derive in finitely many steps that $u$ is an $H^2$–solution of (12.1) on $J_0$ with $u \in W^1_q(J_0, L^p(\mathbb{R}^d))$. \hfill \square

Before discussing global existence, we next derive the conservation laws for the $L^2$–norm and the energy of $H^1$–solutions $u$. We had shown these laws in Remarks 10.4 and 10.5 for more regular solutions. Theorems 12.5 and 13.1 now allow to extend these results to $H^1$–solutions by approximation. We recall that

$$E(v) = \frac{1}{2} ||\nabla v||^2 + \mu \frac{\alpha+1}{\alpha+2} \|v\|_{\alpha+1}^{\alpha+1} = \int_{\mathbb{R}^d} \left( \frac{1}{2} |\nabla v|^2 + \frac{\mu}{\alpha+2} |v|^{\alpha+1} \right) dx$$

for $v \in H^1(\mathbb{R}^d)$ is the energy and that $H^1(\mathbb{R}^d) \hookrightarrow L^{1+\alpha}(\mathbb{R}^d)$ because of $1 < \alpha < \alpha_c$, see (12.4). In particular, $E : H^1(\mathbb{R}^d) \to \mathbb{R}$ is (real) continuously differentiable by Corollary 9.3.
Theorem 13.2. Let \( 1 < \alpha \leq \alpha_c \), \( \mu \in \{-1, 1\} \), \( u_0 \in H^1(\mathbb{R}^d) \) and let \( u \) be the corresponding maximal \( H^1 \)–solution of (13.1) on \( J(u_0) \). We then have

\[
\|u(t)\|_2 = \|u_0\|_2 \quad \text{and} \quad E(u(t)) = E(u_0) \quad \text{for all} \ t \in J(u_0).
\]

Proof. 1) Take \( u_{0,n} \in H^2(\mathbb{R}^d) \) that converge to \( u_0 \) in \( H^1(\mathbb{R}^d) \). Let \( u_n \) be the \( H^2 \)–solution of (13.1) with the initial value \( u_{0,n} \), see Theorem 13.1. Fix a compact interval \( J \subseteq J(u_0) \) with \( 0 \in J \). Theorem 12.5 yields that \( J \subseteq J(u_{0,n}) \) for all \( n \) larger than some \( n_0 \) and that \( u_n(t) \to u(t) \) in \( H^1(\mathbb{R}^d) \) as \( n \to \infty \) for all \( t \in J \). Remark 10.4 now says that \( \|u_n(t)\|_2 = \|u_{0,n}\|_2 \) for all \( t \in J \) and \( n \geq n_0 \). Letting \( n \to \infty \), we derive the first assertion.

2) Since the energy \( E \) is continuous on \( H^1(\mathbb{R}^d) \), the second assertion will follow as the first one if we can show that \( E(u(t)) = E(u_0) \) for all \( t \in J(u_0) \) and \( u_0 \in H^2(\mathbb{R}^d) \). Unfortunately, in Remark 10.5 we needed \( u \in C^1(J(u_0), H^1(\mathbb{R}^d)) \cap C(J(u_0), H^2(\mathbb{R}^d)) \) instead of the available regularity \( u \in C^1(J(u_0), L^2(\mathbb{R}^d)) \cap C(J(u_0), H^2(\mathbb{R}^d)) \). We handle the two terms of the energy in different ways, assuming that \( u_0 \in H^2(\mathbb{R}^d) \) with corresponding \( H^2 \)–solution \( u(J(u_0)) \).

i) We show directly that the function \( t \mapsto \frac{1}{2} \|\Delta u(t)\|_2^2 \) is continuously differentiable on \( J(u_0) \) with derivative \( -\text{Re}(\Delta u_\bar{\tau})_{L^2} \). In fact, let \( t, t + h \in [a, b] \subseteq J(u_0) \). Integrating by parts, we compute

\[
D_h := \frac{1}{2} \|\Delta u(t + h)\|_2^2 - \frac{1}{2} \|\Delta u(t)\|_2^2 + \text{Re}(\Delta u(t)|\bar{\tau})_{L^2} \ h = \text{Re} D_h
\]

\[
= -\frac{1}{2} \text{Re} \int_{\mathbb{R}^d} \Delta u(t + h)\bar{\tau}(t + h) \ dx + \frac{1}{2} \text{Re} \int_{\mathbb{R}^d} \Delta u(t)\bar{\tau}(t) \ dx
\]

\[
+ \text{Re} \int_{\mathbb{R}^d} \Delta u(t)\bar{\tau}'(t)h \ dx
\]

\[
= -\frac{1}{2} \text{Re} \int_{\mathbb{R}^d} (\Delta u(t + h) - \Delta u(t))(\bar{\tau}(t + h) - \bar{\tau}(t)) \ dx
\]

\[
- \text{Re} \int_{\mathbb{R}^d} \Delta u(t)(\bar{\tau}(t + h) - \bar{\tau}(t) - \bar{\tau}'(t)h) \ dx
\]

using also the symmetry of \( \Delta \) on \( H^2(\mathbb{R}^d) \) and \( \text{Re} \ z = \text{Re} \bar{\tau} \) for \( z \in \mathbb{C} \). Observe that \( \bar{\tau} : [a, b] \to L^2(\mathbb{R}^d) \) is Lipschitz with constant \( c := \sup_{a \leq \tau \leq b} \|u'(\tau)\|_2 \). For \( h \neq 0 \) we thus derive

\[
\frac{1}{2} D_h \leq \frac{c}{2} \|\Delta u(t + h) - \Delta u(t)\|_2 + \|\Delta u(t)\|_2 \|\frac{1}{h}(u(t + h) - u(t)) - u'(t)\|_2
\]

\[
\to 0 \quad \text{as} \ h \to 0.
\]

ii) The second summand in the energy is differentiated by means of an approximation of the nonlinearity. To this aim, we fix a function \( \psi \in C^1_0(\mathbb{R}) \) such that \( \psi(r) = \frac{1}{1+r} r^{1+\alpha} \) for \( 0 \leq r \leq 1 \) as well as \( 0 \leq \psi(r) \leq \frac{1}{1+r} r^{1+\alpha} \) and \( 0 \leq \psi'(r) \leq r^{\alpha} \) for all \( r \geq 1 \). We then set \( \phi_n(z) = n^{1+\alpha} \psi(\frac{1}{n}|z|) \) for \( z \in \mathbb{R}^2 \) and \( G_n(v) = \mu \int_{\mathbb{R}^d} \phi_n(v) \ dx \) for \( v \in L^2(\mathbb{R}^d) \). Lemma 9.1 shows that \( G_n : L^2(\mathbb{R}^d) \to \mathbb{R} \) is (real) continuously differentiable with

\[
G_n'(v)w = \mu \int_{\mathbb{R}^d} \nabla \phi_n(v) \cdot w \ dx
\]
for $v, w \in L^2(\mathbb{R}^d)$. As a result,
\[
\frac{d}{dt} G_n(u(t)) = \mu \int_{\mathbb{R}^d} \nabla \phi_n(u(t)) \cdot u'(t) \, dx
\]
for $t \in J(u_0)$. We next note that $\phi_n(u(t))$ tends to $\frac{1}{1+\alpha}|u(t)|^{\alpha+1}$ and $\nabla \phi_n(u(t)) \cdot u(t)$ tends to $|u(t)|^{\alpha-1} \text{Re}(u(t)\overline{\pi}(t))$, pointwise as $n \to \infty$. Moreover,
\[
|\phi_n(z)| \leq \frac{1}{1+\alpha} n^{1+\alpha} |z|^{1+\alpha} = \frac{1}{1+\alpha} |z|^{1+\alpha}, \quad |\nabla \phi_n(z)| \leq \frac{n^{1+\alpha}}{n^\alpha} \frac{|z|}{n |z|} |z|^\alpha = |z|^\alpha
\]
for $z \in \mathbb{R}^d$. Recall that $u(t) \in H^2(\mathbb{R}^d) \to L^{1+\alpha}(\mathbb{R}^d) \cap L^{2\alpha}(\mathbb{R}^d)$ by Sobolev’s embedding (5.5) and that $u'(t) \in L^2(\mathbb{R}^d)$. With the majorants $\frac{1}{1+\alpha} |u(t)|^{\alpha+1}$ and $|u(t)|^{\alpha} |u'(t)|$, dominated convergence yields
\[
G_n(u(t)) \to \frac{\mu}{1+\alpha} \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} \, dx,
\]
\[
\frac{d}{dt} G_n(u(t)) = \mu \int_{\mathbb{R}^d} \nabla \phi_n(u(t)) \cdot u'(t) \, dx \to \mu \text{Re} \int_{\mathbb{R}^d} |u(t)|^{\alpha-1} u(t) \overline{\pi}(t) \, dx
\]
as $n \to \infty$ for each $t \in \mathbb{R}$. Moreover, $G_n(u(t))$ and $\frac{d}{dt} G_n(u(t))$ are uniformly bounded for $n \in \mathbb{N}$ and $t$ in compact subsets of $J(u_0)$.

iii) We next define the “approximative energy”
\[
E_n(v) = \frac{1}{2} |||\nabla v|||^2 + G_n(v)
\]
for $n \in \mathbb{N}$ and $v \in H^1(\mathbb{R}^d)$. The results in i) and ii) then show that
\[
E_n(u(t)) \to E(u(t)),
\]
\[
\frac{d}{dt} E_n(u(t)) \to \text{Re} \int_{\mathbb{R}^d} (-\Delta u(t) + \mu |u(t)|^{\alpha-1} u(t)) \overline{\pi}(t) \, dx
\]
\[
= \text{Re} \int_{\mathbb{R}^d} i u'(t) \overline{\pi}(t) \, dx = 0
\]
as $n \to \infty$ for all $t \in J(u_0)$, because $u$ solves (13.1). Since $E_n(u)$ and $\frac{d}{dt} E_n(u)$ are locally bounded, the above limits also hold in $L^1(J)$ for each open bounded interval with $J \subseteq J(u_0)$. Hence, $E(u) \in W^1_1(J)$ with vanishing derivative and $E(u)$ is constant.

In several cases the above conservation laws allow to bound the $H^1$–norm of a solution. Theorem 12.5 shows that this norm must explode in finite time if we do not have global existence. This line of arguments leads to our final theorem. In assertion (a) we only use that in the defocusing case the energy plus the $L^2$–norm dominate the norm in $H^1(\mathbb{R}^d)$. In the focusing case the second summand of $E(u(t))$ is negative and has to be controlled by the first part of $E(u(t))$ and $||u(t)||^2$. This can be done if either $\alpha$ or $u_0$ is small.

**Theorem 13.3.** Let $\mu \in \{-1, 1\}$, $\alpha < \frac{d+2}{(d-2)+} = \alpha_e$, $u_0 \in H^1(\mathbb{R}^d)$ and let $u$ be the corresponding maximal $H^1$–solution of (13.1) on $J(u_0)$. Then the following assertions hold.

(a) If $\mu = 1$, then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1(\mathbb{R}^d)$.

(b) If $\mu = -1$ and $\alpha < \alpha + 1 + \frac{4}{d}$, then $J(u_0) = \mathbb{R}$ for all $u_0 \in H^1(\mathbb{R}^d)$.  

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(c) There are numbers \( \rho, \kappa > 0 \) such that \( J(u_0) = \mathbb{R} \) and \( \sup_{t \in \mathbb{R}} \|u(t)\|_{1,2} \leq \kappa \) if \( \|u_0\|_{1,2} \leq \rho \).

**Proof.** (a) If \( \mu = 1 \), Theorem 13.2 yields that
\[
\|u(t)\|^2 \leq 2E(u(t)) = \|u_0\|^2 + 2E(u_0)
\]
for all \( t \in J(u_0) \). From the blow-up criterion in Theorem 12.5 (c) we thus deduce \( J(u_0) = \mathbb{R} \).

(b) Let \( 1 < \alpha < 1 + \frac{2}{d} \) and \( \mu = -1 \). We consider \( d \geq 3 \), the proof for \( d = 1,2 \) is similar. The proof of Sobolev’s embedding even yields that
\[
\|v\|_{\frac{2d}{d-2}} \leq c_d \|
abla v\|_2
\]
for \( v \in H^1(\mathbb{R}^d) \) and a constant \( c_d \) only depending on \( d \), see (D.15) in Appendix D. To use this extra integrability of \( u(t) \), we note that
\[
\frac{1}{\alpha+1} = \frac{1}{2} + \frac{\theta}{d-2} \quad \text{for} \quad \theta = \frac{d}{2} - \frac{d}{\alpha+1} \in (0,1).
\]
The interpolation inequality and (13.9) then imply
\[
\|v\|_0^{\alpha+1} \leq \|v\|_2^{\alpha(1-\theta)/(\alpha+1)} \|v\|_2^{\theta(\alpha+1)} \leq c_d^{\theta(\alpha+1)} \|v\|_2^{\alpha+1-d\frac{\alpha+1}{2d}} \|\nabla v\|_{d\frac{\alpha+1}{2d}}
\]
for all \( v \in H^1(\mathbb{R}^d) \). We have \( \beta := \frac{4}{d(\alpha-1)} > 1 \) by the assumption on \( \alpha \). Young’s inequality with exponents \( \beta \) and \( \beta' \) now leads to
\[
\frac{1}{\alpha+1} \|v\|_0^{\alpha+1} \leq \frac{1}{4} \|\nabla v\|_2^2 + c \|v\|_2^{\beta(\alpha+1-d\frac{\alpha+1}{2d})}
\]
for a constant \( c \) only depending on \( \alpha \) and \( d \). Denoting the last summand by \( k(\|v\|_2) \), we infer from Theorem 13.2 that \( \|u(t)\|^2_2 = \|u_0\|^2_2 \) and
\[
E(u_0) = E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_2^2 - \frac{1}{\alpha+1} \|u(t)\|_{\alpha+1}^{\alpha+1} \geq \frac{1}{4} \|\nabla u(t)\|_2^2 - k(\|u_0\|_2)
\]
for all \( t \in J(u_0) \). Therefore, \( \|u(t)\|_{1,2}^2 \leq 4E(u_0) + 4k(\|u_0\|_2) + \|u_0\|_2^2 \) for all \( t \in J(u_0) \), and as before it follows that \( J(u_0) = \mathbb{R} \).

(c) Let \( \mu \in \{-1,1\} \). This part relies on the observation that by Sobolev’s embedding the “nonlinear” part of the energy can be bounded by
\[
\frac{1}{1+\alpha} \|u(t)\|_{\alpha+1}^{\alpha+1} \leq c \|u(t)\|_{1,2}^{\alpha-1} \|u(t)\|_{1,2}^2.
\]
So if \( \|u_0\|_{1,2} \) is small, one can absorb this term by the other part of the energy and \( \|u(t)\|_2^2 \) as long as \( \|u(t)\|_{1,2} \) stays under a certain constant \( \gamma \). Choosing a suitable \( \gamma > \|u_0\|_{1,2} \), one then sees by a contradiction argument that actually \( \|u(t)\|_{1,2} \leq \gamma \) for all \( t \) and the assertion will follow.

To make this precise, we note that the conservation laws and Sobolev’s embedding yield
\[
\frac{1}{2} \|u(t)\|_{1,2}^2 = \frac{1}{2} \|u_0\|^2_2 + E(u(t)) - \frac{\mu}{\alpha+1} \|u(t)\|_{\alpha+1}^{\alpha+1} \leq \frac{1}{2} \|u_0\|_2^2 + E(u_0) + c_0 \|u(t)\|_{1,2}^{\alpha-1} \|u(t)\|_{1,2}^2,
\]
(13.10)

\(^1\)Such estimates are called Gagliardo-Nirenberg inequalities.
where \( c_0 = C_{S_0}^{\alpha + 1} \frac{1}{1+\alpha} \) with Sobolev’s constant \( C_{S_0} \) from (12.4). We set \( \gamma = (4c_0)^{\frac{1}{1+\alpha}} \) and take any \( \rho \in (0, \gamma) \). Let \( \|u_0\|_{1,2} \leq \rho \). We now define
\[
t_0 = \sup \left\{ t \in (0, t^+(u_0)) \mid \|u(s)\|_{1,2} \leq \gamma \text{ for all } s \in [0, t] \right\}.
\] (13.11)

Observe that \( t_0 \in (0, t^+(u_0)) \). Let \( 0 \leq t < t_0 \). The estimate (13.10), the choice of \( \gamma \) and Sobolev’s embedding (12.4) next imply that
\[
\frac{1}{2}\|u(t)\|_{1,2}^2 \leq \frac{1}{2}\|u_0\|_{2}^2 + E(u_0) + \frac{1}{2}\|u(t)\|_{1,2}^2,
\]
\[
\|u(t)\|_{1,2}^2 \leq 2\|u_0\|_{2}^2 + 2\|\nabla u_0\|_{2}^2 + \frac{1}{\alpha+1}\|u_0\|_{\alpha+1}^\alpha \leq c_1(\rho^2 + \rho^{\alpha+1}),
\] (13.12)
where \( c_1 = \max\{2, \frac{4}{1+\alpha}C_{S_0}^{1+\alpha}\} \). We now fix \( \rho \in (0, \gamma) \) such that \( c_1(\rho^2 + \rho^{\alpha+1}) \leq \frac{\gamma^2}{16} \). If \( t_0 < t^+(u_0) \), then \( u(t) \to u(t_0) \) in \( H^1(\mathbb{R}^d) \) as \( t \to t_0 \) so that (13.11) yields \( \|u(t_0)\|_{1,2} = \gamma \), but (13.12) leads to the contradiction \( \|u(t_0)\|_{1,2} \leq \frac{\gamma}{2} \).

Hence, \( t_0 = t^+(u_0) \) and from (13.12) we derive that \( \|u(t)\|_{1,2} \leq \frac{\gamma}{2} \) for all \( t \in [0, t^+(u_0)) \). Theorem 12.5 (c) now implies \( t^+(u_0) = \infty \), as asserted.

Similarly one treats negative times. \( \Box \)

The proof of Theorem 13.3 (c) leads to a corollary concerning stability.

**Corollary 13.4.** Let \( \mu \in \{-1, 1\} \) and \( 1 < \alpha < \alpha_c \). Then the solution \( u_* = 0 \) of (13.3) is Lyapunov stable in \( H^1(\mathbb{R}^d) \), i.e.,
\[
\forall \varepsilon > 0 \exists \delta > 0 \forall u_0 \in \overline{B}_{H^1}(0, \delta) : J(u_0) = \mathbb{R} \text{ and } \|u(t; u_0)\|_{1,2} \leq \varepsilon \text{ for all } t \in \mathbb{R}.
\]

**Proof.** Let \( c_0 \) and \( c_1 \) be given as in the proof of Theorem 13.3 (c). Set \( \varepsilon_0 = \frac{1}{2}(4c_0)^{-\frac{1}{1+\alpha}} \) and take any \( \varepsilon \in (0, \varepsilon_0] \). Choose \( \delta \in (0, 2\varepsilon) \) such that \( c_1(\delta^2 + \delta^{\alpha+1}) \leq \varepsilon^2 \). Take \( u_0 \in \overline{B}_{H^1}(0, \delta) \). Define \( t_0 \) as in (13.11) with \( \gamma \) replaced by \( 2\varepsilon \). The estimate (13.12) still holds and implies the assertion. \( \Box \)

We conclude this lecture with a few remarks about further results on global existence and blow-up.

Global existence holds in the defocusing case \( \mu = 1 \) also if \( \alpha = \frac{d+2}{d-2} \) and \( d \geq 3 \). This result is far beyond the scope of these lectures, see Chapter 5 of [Tao06] for an extended survey.

For \( \alpha \in \left[ 1 + \frac{4}{d}, \alpha_c \right) \) and \( \mu = -1 \), Theorem 6.5.4 in [Caz03] establishes blow-up in (13.1) if \( E(u_0) < 0 \) and \( |x| u_0 \in L^2(\mathbb{R}^d) \).\(^2\) (This additional integrability is not needed if \( u_0 \) is spherically symmetric by Theorem 6.5.10 in [Caz03].) One could guess that a negative initial energy is necessary for blow-up. This is not the case as Remark 6.5.8 in [Caz03] gives a blow-up solution with \( E(u_0) > 0 \).

In Example 10.3 we have seen a blow-up solution for \( \alpha = 1 + \frac{4}{d} \) and \( \mu = -1 \). Denote its initial value by \( \varphi \) and consider (13.1) with \( \mu = -1 \) and \( \alpha = 1 + \frac{4}{d} \). If \( \|u_0\|_2 < \|\varphi\|_2 \), then \( J(u_0) = \mathbb{R} \) by Theorem 6.6.1 in [Caz03]. So in the borderline case \( \alpha = 1 + \frac{4}{d} \), where global existence starts to fail for \( \mu = -1 \), one has a precise threshold for the occurrence of blow-up solutions.

Let \( \mu = -1 \) and \( \max\{1, \alpha_0\} < \alpha < \alpha_c \), where \( \alpha_0 > 0 \) satisfies \( \alpha_0^2 + (d-2)x_0 = 4 \). Let \( u_0 \in H^1(\mathbb{R}^d) \) with \( |x| u_0 \in L^2(\mathbb{R}^d) \). Set \( \varphi_b(x) = e^{ib|x|^2} u_0(x) \) for \( b > 0 \) and

\(^2\)Recall that in the corrected version of Proposition 9.6 we showed blow-up for a certain nonlinear wave equation if the initial energy is negative and another condition holds.
Let \( x \in \mathbb{R}^d \). Then \( \varphi_b \in H^1(\mathbb{R}^d) \) and there is a number \( b_0 > 0 \) such that \( t^+(\varphi_b) = \infty \) for all \( b \geq b_0 \). (See Theorem 6.3.4 of \( \text{Caz03} \).) Hence, one has global existence (to the right) if the initial value is rapidly oscillating.

**Exercises**

**Exercise 13.1.** Let \( u_0 \in H^1(\mathbb{R}^d) \) and \( u \) be the corresponding \( H^1 \)-solution of (13.1). Define its “momentum” by

\[
p_j(t) = \text{Im} \int_{\mathbb{R}^d} \overline{u(t)} \partial_j u(t) \, dx, \quad t \in J(u_0), \quad j \in \{1, \ldots, d\}.
\]

Show that \( p_j(t) = p_j(0) \) for all \( t \in J(u_0) \). (Hint: Consider first \( u_0 \in H^2(\mathbb{R}^d) \).)

**Exercise 13.2.** Let \( u \) be an \( H^2 \)-solution of (13.1) on \( J = [0, b) \), where \( b \in (0, \infty) \) and \( \alpha = 1 + \frac{4}{d} \). Let \( \gamma \geq 1 \) and set

\[
u_\gamma(t, x) = (1 + \gamma t)^{-\frac{d}{2}} e^{i \frac{\gamma t}{1+\gamma t} x} u \left( \frac{t}{1+\gamma t}, \frac{1}{1+\gamma t} x \right)
\]

for \( t \in \mathbb{R}_+ \) and \( x \in \mathbb{R}^d \). Show that \( u_\gamma \) satisfies (13.1) with \( u_\gamma(0) = e^{i|\cdot|^2} u(0) \).

Further show that

(i) \( \|u_\gamma(t)\|_2 = \|u(s)\|_2, \quad \|u_\gamma(t)\|_{1+\alpha} = (1 + \gamma t)^{-\frac{d}{1+\alpha}} \|u(s)\|_{1+\alpha}, \)

(ii) \( \|\nabla u_\gamma(t)\|_2 = \frac{1}{2+2d} \|(|i\gamma x + 2\nabla|)u(s)\|_2 \)

for all \( t \geq 0 \), where \( s = s(t) = t(1 + \gamma t)^{-1} \) and we assume that \( |x|u(s) \in L^2(\mathbb{R}^d) \) in (ii).

**Exercise 13.3.** Let \( d \geq 3, \alpha = \frac{d+2}{2} \) and \( \mu \in \{-1, 1\} \). Show that there is a radius \( \rho > 0 \) such that (13.1) has an \( H^1 \)-solution in \( E_1(\mathbb{R}) \) for all \( u_0 \in \overline{B}_H(0, \rho) \). (Hint: Use Strichartz’ estimates for the endpoint case \( (q, p) = (2, \frac{2d}{d-2}) \).)
Bibliography


