Center-stable manifolds for standing waves of the Klein-Gordon equation

In this project, the main object of consideration will be the Klein-Gordon (KG) equation

\[ u_{tt}(t, x) - \Delta u(t, x) + u(t, x) - |u(t, x)|^{p-1}u(t, x) = 0, \quad (t, x) \in \mathbb{R}^1 \times \mathbb{R}^d \]

It is well-known that there exists a one parameter family of standing wave solutions (sometimes referred to as solitons) in the form \( e^{i\omega t} \phi_\omega(x) \), \( |\omega| \in [0, 1) \), \( \phi_\omega > 0 \), which satisfy

\[ -\Delta \phi + (1 - \omega^2)\phi = \phi(x)^p \]

Some of these waves are linearly stable (depending on the values of \( \omega, d \)), while others are unstable. We will be interested in the behavior of the solutions of KG, when the data is in a vicinity of an unstable soliton. While one clearly expects a generic data to deviate from the soliton, there is an important subset of this data set that actually produce solutions that converge to the soliton. This is the notion of the center-stable manifold. In a recent paper, [2], this result was proved in high dimensions \( d \geq 2 \). The one dimensional result (that is for \( d = 1 \)) appeared in [1].

The construction of the center-stable manifold involves various tools developed in the lectures, like functional analytic methods and fixed point theorems, Strichartz estimates (but this time for the Klein-Gordon equation with potentials) etc. As an interesting aside in all of this is the role of the spatial dimension, leading to the somewhat counterintuitive conclusion that for dispersive equations things are easier in higher dimensions. This is basically due to the lower rate of dispersion in low dimensions, which presents another layer of difficulty in the analysis of these models in \( d = 1, 2 \).

References