LECTURE 2

Strongly continuous semigroups and their generators

In the first part of the Internet Seminar we investigate the linear evolution equation (or more precisely, the initial value or Cauchy problem)

\[ u'(t) = Au(t), \quad t \geq 0, \quad u(0) = u_0, \]

(2.1)
on a Banach space \( X \). Here \( A \) is a given linear operator with domain \( D(A) \) and the initial value \( u_0 \in X \) (often \( u_0 \in D(A) \)). We want to develop a systematic theory for (2.1) which will also be the basis for our study of semilinear problems. As explained in Lecture 1, the solution of (2.1) will be given by \( u(t) = T(t)u_0 \) for an operator semigroup \( (T(t))_{t \geq 0} = T(\cdot) \) on \( X \). Our analysis starts with the study of such semigroups. For each semigroup we introduce its generator \( A \) as the derivative of \( t \to T(t) \) at \( t = 0 \), roughly speaking. We next investigate some of their basic properties, solve problem (2.1) for the generator \( A \), and study a simple but instructive class of examples, the translation semigroups.

In the following lectures, we will tackle the more difficult problem to characterize those (given) operators \( A \) which are generators of a strongly continuous semigroup.

A few words about our notation: Throughout we assume that \((X, \| \cdot \|_X)\) is a complex Banach space with \( X \neq \{0\} \), where we mostly write \( \| \cdot \| \) instead of \( \| \cdot \|_X \), if no confusion is to be expected. By \( \mathcal{B}(X,Y) \) we denote the space of all bounded linear operators from \( X \) into another Banach space \( Y \), setting \( \mathcal{B}(X) = \mathcal{B}(X,X) \).

Mostly, the operator norm is also designated by \( \| \cdot \| \). Further, \( X^* \) is the dual space of \( X \) and \( I \) is the identity map on \( X \). We write \( (x,x^*) := x^*(x) \) for \( x^* \in X^* \) and \( x \in X \). The scalar product on a Hilbert space \( H \) is denoted by \( (x|y)_H \), or simply \( (x|y) \), for \( x,y \in H \). By \( 1_M \) we designate the characteristic function of a set \( M \). We put \( \mathbb{R}_+ = [0,\infty) \) and \( \mathbb{R}_- = (-\infty,0] \).

For \( M \subseteq \mathbb{R}^d \) we write \( C(M,X) \) for the vector space of continuous functions \( f : M \to X \). We use the subspaces

\[
C_b(M,X) = \{ f \in C(M,X) \mid f \text{ is bounded} \}, \\
C_0(M,X) = \{ f \in C(M,X) \mid f(s) \to 0 \text{ as } |s| \to \infty \text{ and as } s \to \partial M \setminus M \}, \\
C_c(M,X) = \{ f \in C(M,X) \mid \text{supp } f \subseteq M \text{ is compact} \},
\]

where \( C_b(M) := C_b(M,\mathbb{C}) \) etc. Here \( \text{supp } f \) is the closure in \( \mathbb{R}^d \) of the set \[ \{ s \in M \mid f(s) \neq 0 \} \]. Below we will repeat these definitions in specific cases. The spaces \( C_b(M,X) \) and \( C_0(M,X) \) are always equipped the sup–norm \( \| f \|_\infty := \sup_{s \in M} \| f(s) \| \) and then become Banach spaces.

We employ analogous notations for spaces of differentiable functions. For instance, \( C_0^1([0,1]) = \{ f \in C^1([0,1]) \mid f, f' \in C_0([0,1]) \} \).
If \( M \) is a Borel subset of \( \mathbb{R}^d \) and \( p \in [1, \infty] \), we write \( L^p(M) \) for the usual Lebesgue space of complex valued functions with respect to the Lebesgue measure on \( M \) and endow it with the \( p \)-norm, given by \( \|f\|_p^p := \int_M |f|^p \, dx \) if \( p < \infty \) and by the essential supremum for \( p = \infty \). If \( M = (a, b) \subseteq \mathbb{R} \), we write \( C_0(a, b) \), \( L^p(a, b) \) and so on.

**Definition 2.1.** A map \( T(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X) \) is called strongly continuous operator semigroup or just \( C_0 \)-semigroup if the following conditions are fulfilled:

(a) \( T(0) = I \) and we have \( T(t+s) = T(t)T(s) \) for all \( t, s \geq 0 \).

(b) For each \( x \in X \) the orbit, defined as the map

\[
T(\cdot)x : \mathbb{R}_+ \to X, \quad t \mapsto T(t)x,
\]

is continuous.

The generator \( A \) of \( T(\cdot) \) is given by setting

\[
D(A) := \{ x \in X \mid \text{the limit } \lim_{t \to 0^+} \frac{1}{t}(T(t)x - x) \text{ exists in } X \}
\]

and defining

\[
Ax := \lim_{t \to 0^+} \frac{1}{t}(T(t)x - x)
\]

for \( x \in D(A) \). We also say that \( A \) generates \( T(\cdot) \).

If one replaces throughout in this definition \( \mathbb{R}_+ \) by \( \mathbb{R} \) and “\( t \to 0^+ \)” by “\( t \to 0 \)”, one obtains the concept of a \( C_0 \)-group with generator \( A \).

Property (a) in Definition 2.1 is called the semigroup law and (b) is the strong continuity. We point out that the semigroup does not need to be continuous with respect to the operator norm, cf. Example 2.6. Observe that the above definition directly implies that \( D(A) \) is a linear subspace of \( X \) and \( A \) is a linear map in \( X \) which is uniquely determined by the semigroup.

Let \( T(\cdot) \) be a \( C_0 \)-semigroup. The semigroup operators then commute since

\[
T(t)T(s) = T(t+s) = T(s+t) = T(s)T(t)
\]

(2.2)

holds for all \( t, s \geq 0 \). By induction, we obtain

\[
T(nt) = T\left( \sum_{j=1}^n t \right) = \prod_{j=1}^n T(t) = T(t)^n
\]

(2.3)

for all \( n \in \mathbb{N} \) and \( t \geq 0 \). If \( T(\cdot) \) is even a \( C_0 \)-group, it satisfies

\[
T(t)T(-t) = T(0) = I = T(-t)T(t)
\]

(2.4)

for all \( t \in \mathbb{R} \). Hence, \( T(t) \) is invertible with inverse \( T(-t) \) for every \( t \in \mathbb{R} \).

We first look at the finite dimensional situation, which is rather simple since here one can construct the semigroup from the given operator \( A \) by a power series. This does not work for unbounded \( A \).

**Example 2.2.** Let \( X = \mathbb{C}^d \), \( A \in \mathcal{B}(X) \cong \mathbb{C}^{d \times d} \) and set

\[
T(t) := e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n
\]

for \( t \in \mathbb{R} \). It is known from analysis courses that the series converges in \( \mathcal{B}(X) \), \( T(\cdot) \) satisfies (a) in Definition 2.1 and \( T(\cdot) \) is continuously differentiable even in
Lemma 2.3. Assertion (b) follows from (c) by the Banach-Steinhaus theorem. The analogous equivalences hold in the group case.

Example 2.6. The notation \( \omega \) is called the growth bound of \( \mathcal{B}(X) \) with \( \frac{d}{dt} e^{tA} = Ae^{tA} \) for all \( t \in \mathbb{R} \). In particular, \( T(\cdot) \) is a \( C_0 \)-group with generator \( A \). Moreover, for any given \( u_0 \in X \) the function \( u : \mathbb{R}_+ \to X \) defined by \( u(t) = e^{tA}u_0 \) solves the linear ordinary differential equation 
\[
    u'(t) = Au(t), \quad t \in \mathbb{R}, \quad u(0) = u_0.
\]
The same results hold for any bounded linear operator \( A \) on a Banach space \( X \), see Exercise 2.1.

The simple Definition 2.1 has many astonishing consequences. We first observe that every \( C_0 \)-semigroup is exponentially bounded. This fact then leads to the subsequent basic definition.

Lemma 2.3. Let \( T(\cdot) \) be a \( C_0 \)-semigroup. Then there are \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|T(t)\| \leq Me^{\omega t} \) holds for all \( t \geq 0 \).

Proof. By strong continuity, each orbit \( T(\cdot)x \) is bounded on \([0, 1]\). Hence, there is a constant \( c > 0 \) with \( \|T(\tau)\| \leq c \) for all \( \tau \in [0, 1] \) due to the principle of uniform boundedness. Let \( t \geq 0 \). Take \( n \in \mathbb{N}_0 \) and \( \tau \in [0, 1) \) with \( t = n + \tau \). Setting \( \omega := \log c \), we deduce from Definition 2.1 (a) and (2.3) that
\[
    \|T(t)\| = \|T(\tau)T(n)\| = \|T(\tau)T(1)^n\| \leq c \cdot c^n = e^{(n+1)\log c} = e^{\omega} \cdot e^{(1-\tau)\log c} \leq e^{\max\{\log c, 0\}}e^{\omega}.
\]

Definition 2.4. Let \( T(\cdot) \) be a \( C_0 \)-semigroup with generator \( A \). Then
\[
    \omega_0(T) := \omega_0(A) := \inf \left\{ \omega \in \mathbb{R} \mid \exists M_\omega \geq 1 : \|T(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0 \right\}
\]
is called the growth bound of \( T(\cdot) \).

Lemma 2.3 says that \( \omega_0(T) < \infty \). It may happen that \( \omega_0(T) = -\infty \), see Example 2.6. The notation \( \omega_0(A) \) will be justified in the next lecture where we show that an operator \( A \) can generate at most one \( C_0 \)-semigroup. We point out that in general the infimum in Definition 2.4 is not a minimum even in the matrix case. For instance, for \( X = \mathbb{C}^2 \) (endowed with the 1-norm) and \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), we have \( T(t) = e^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \), so that \( \|T(t)\| = t + 1 \) for \( t \geq 0 \), while \( \omega_0(T) = 0 \).

The next lemma often helps to verify the strong continuity of an operator semigroup.

Lemma 2.5. Let \( T(\cdot) : \mathbb{R}_+ \to \mathcal{B}(X) \) be a map satisfying condition (a) in Definition 2.1. Then the following assertions are equivalent.

(a) \( T(\cdot) \) is strongly continuous (and thus a \( C_0 \)-semigroup).
(b) It holds \( \lim_{t \to 0^+} T(t)x = x \) for all \( x \in X \).
(c) There are a number \( t_0 > 0 \) and a dense subspace \( D \subseteq X \) such that \( \sup_{0 \leq t \leq t_0} \|T(t_0)\| < \infty \) and \( \lim_{t \to 0^+} T(t)x = x \) for all \( x \in D \).

The analogous equivalences hold in the group case.

Proof. The implication “(a)⇒(c)” is an immediate consequence of Lemma 2.3. Assertion (b) follows from (c) by the Banach-Steinhaus theorem.
Indeed, consider $f$ satisfying
\[ ||f|| \leq ||T|| \cdot ||T(h)x - x||, \]
where the right hand side of this inequality converges to 0 as $h$ tends to 0. In the case that $h \in (-t, 0]$, we note that Lemma 2.3 yields
\[ ||T(t + h)|| \leq Me^{\omega(t+h)} \leq Me^{\omega|t|} \]
for some constants $M \geq 1$ and $\omega \in \mathbb{R}$. As a result,
\[ ||T(t + h)x - T(t)x|| = ||T(t + h)(x - T(-h)x)|| \leq Me^{\omega|t|} \cdot ||x - T(-h)x|| \to 0 \]
as $h \to 0^-$, and (a) holds. The assertions about groups are shown similarly. □

In the above lemma the implication “(c)$\Rightarrow$(a)” can fail if one omits the boundedness assumption (cf. Exercise 1.5.9(4) in [EN99]). We now examine a basic class of examples for $C_0$-semigroups, the translation semigroups. They are given by an explicit formula (which is a rare exception) and are thus very convenient to illustrate various aspects of the theory. Their generators will be determined in the next lecture.

**Example 2.6** (Left translation semigroups on $\mathbb{R}$ and $[0,1]$).

(a) Let $X = C_0(\mathbb{R}) = \{ f \in C(\mathbb{R}) \mid f(s) \to 0 \text{ as } |s| \to \infty \}$ and $T(\cdot)$ be given by
\[ (T(t)f)(s) := f(s + t) \quad \text{for } t \in \mathbb{R}, \; f \in X, \; s \in \mathbb{R}. \]
We claim that $T(\cdot)$ is a $C_0$-group on $X$. Clearly, $T(0) = I$ and $T(t)$ is a linear isometry on $X$ so that $||T(t)|| = 1$. We further obtain
\[ T(t)T(r)f = (T(r)f)(\cdot + t) = f(\cdot + t + r) = T(t + r)f \]
for all $f \in X$ and $r, t \in \mathbb{R}$. Hence, $T(t)T(r) = T(t + r)$. We employ Lemma 2.5 to verify the strong continuity. For $f \in C_c(\mathbb{R})$ the function $T(t)f$ converges uniformly to $f$ as $t$ tends to 0 since $f$ is uniformly continuous. It thus remains to check $C_c(\mathbb{R}) = C_0(\mathbb{R})$. For each $n \in \mathbb{N}$ take a “cut–off” function $\varphi_n \in C(\mathbb{R})$ satisfying $\varphi_n = 1$ on $[-n,n]$, $0 \leq \varphi_n \leq 1$ and $\operatorname{supp} \varphi_n \subseteq (-n-1, n+1)$. For $f \in C_0(\mathbb{R})$, we then have $\varphi_n f \in C_c(\mathbb{R})$ and
\[ ||f - \varphi_n f||_\infty = \sup_{|s| \geq n} |(1 - \varphi_n(s))f(s)| \leq \sup_{|s| \geq n} |f(s)| \to 0 \]
as $n \to \infty$. We now conclude that $T(\cdot)$ is a $C_0$-group by means of Lemma 2.5.

The same assertions hold for $X = L^p(\mathbb{R})$ with $1 \leq p < \infty$ by similar arguments, see Exercise 2.2.

In contrast to these results, $T(\cdot)$ is not strongly continuous on $X = L^\infty(\mathbb{R})$. Indeed, consider $f = 1_{[0,1]}$ and observe that
\[ T(t)f(s) = 1_{[0,1]}(s + t) = \begin{cases} 1, & s + t \in [0,1] \\ 0, & s + t \notin [0,1] \end{cases} = 1_{[-t,1-t]}(s) \]
for $s, t \in \mathbb{R}$. Thus, $||T(t)f - f||_\infty = 1$ for every $t \neq 0$.

In addition, $T(\cdot)$ is not continuous as a $\mathcal{B}(X)$-valued function for $X$ being $L^p(\mathbb{R})$ (see Exercise 2.2) or $C_0(\mathbb{R})$. In fact, for $X = C_0(\mathbb{R})$ consider for each $n \in \mathbb{N}$
We show that according to Lemma 2.5, whenever 
lim_{s \to 1} f(s) = 0 \implies \text{the unique solution of (2.5). Moreover,}
T(\cdot) is not bijective. As in (a) one sees that 
\omega_0(T) = \infty. Let t, r \geq 0 and s \in [0, 1). We then obtain
\[
(T(t)T(r)f)(s) = \begin{cases} 
(T(r)f)(s + t), & \text{if } s + t < 1, \\
0, & \text{else}, \\
\end{cases}
= \begin{cases} 
f(s + t + r), & \text{if } s + t < 1, \quad s + t + r < 1, \\
0, & \text{else}, \\
\end{cases}
= (T(t + r)f)(s).
\]
Hence, T(\cdot) is a semigroup (which cannot be extended to a group since 
e.g. \(T(1) = 0\) is not bijective). As in (a) one sees that \(C_c([0, 1]) = \{f \in C([0, 1]) | \exists b f \in (0, 1) : \text{supp } f \subseteq [0, b f]\}\) is a dense subspace of X. For 
f \in C_c([0, 1]) and t \in (0, 1 - b f), we compute
\[
T(t)f(s) - f(s) = \begin{cases} 
f(s + t) - f(s), & s \in [0, 1 - t), \\
0, & s \in [1 - t, 1) \subseteq [b f, 1], \\
\end{cases}
\]
and deduce \(\lim_{t \to 0} \|T(t)f - f\|_{\infty} = 0\) using the uniform continuity of f. 
According to Lemma 2.5, T(\cdot) is a \(C_0\)-semigroup on X. ♦

We next state the solution concept for equation (2.1).

**Definition 2.7.** Let A be a linear operator on X with domain \(D(A)\) and let 
x \in D(A). We say a function \(u : \mathbb{R}_+ \to X\) solves the Cauchy problem
\[
u'(t) = Au(t), \quad t \geq 0, \quad u(0) = x, \quad (2.5)
\]
if \(u \in C^1(\mathbb{R}_+, X)\) satisfies \(u(t) \in D(A)\) for all \(t \geq 0\) and fulfills (2.5).

We next show that if A generates a \(C_0\)-semigroup, then the semigroup gives the unique solution of (2.5). Moreover, \(T(t)\) and A commute on \(D(A)\).

**Proposition 2.8.** Let A generate the \(C_0\)-semigroup \(T(\cdot)\) and \(x \in D(A)\). 
Then \(T(t)x \in D(A), \ AT(t)x = T(t)Ax\) for all \(t \geq 0\) and the function
\[
u : \mathbb{R}_+ \to X, \quad t \mapsto T(t)x,
\]
is the unique solution of (2.5).
Proof. 1) Let \( t \geq 0, h > 0 \) and \( x \in \text{D}(A) \). We obtain
\[
\frac{1}{h}(T(t+h)x - T(t)x) = \frac{1}{h}(T(h)x - T(0)x) = \frac{1}{h}(T(h)x - x) \to T(t)Ax
\]
as \( h \to 0 \). The very definition of \( A \) yields \( T(t)x \in \text{D}(A) \) and \( AT(t)x = T(t)Ax \). Moreover, \( T(\cdot)x \) is differentiability from the right. Let \( 0 < h < t \). It further holds
\[
\frac{1}{h}(T(t-h)x - T(t)x) = \frac{1}{h}(T(h)x - x) \to T(t)Ax
\]
as \( h \to 0 \), where we have used Lemma 2.9 below (with \( S(t,h) = T(t-h) \)). Since \( T(\cdot)x \) is continuous, we have shown that \( T(\cdot)x \in C^1(\mathbb{R}_+, X) \) with derivative \( \frac{d}{dt} T(\cdot)x = AT(\cdot)x \); i.e., \( u \) solves (2.5).

2) Let \( v \) be another solution of (2.5) and \( t > 0 \). Set \( w(s) := T(t-s)v(s) \) for \( s \in [0,t] \). Lemma 2.9 (with \( S(t,s) = T(t-s) \) and \( Y = \text{D}(A) \)) and the first step now imply that
\[
\frac{d}{ds} w(s) = T(t-s)v'(s) - T(t-s)Av(s) = 0,
\]
where the last equality follows from the assumption that \( v \) solves (2.5). So for every \( x^* \in X^* \) the scalar function \( \langle w(\cdot), x^* \rangle \) is differentiable with vanishing derivative and thus constant, which leads to
\[
(T(t), x^*) = \langle w(0), x^* \rangle = \langle w(t), x^* \rangle = \langle v(t), x^* \rangle
\]
for all \( t \geq 0 \) and \( x^* \in X^* \). The Hahn-Banach theorem now yields \( T(\cdot)x = v \). \( \square \)

In general one really needs the extra condition that \( x \in \text{D}(A) \) to obtain a solution of (2.5). For instance, if \( f \in C_0(\mathbb{R}) \setminus C^1(\mathbb{R}) \), then the orbit \( T(\cdot)f \) of the translation semigroup on \( C_0(\mathbb{R}) \) is not differentiable, cf. Example 2.6. We continue with the lemma used in the previous proof.

Lemma 2.9. Let \( b > a \) be real numbers, \( M = \{(t,s) \in [a,b]^2 \mid t \geq s\} \), \( S : M \to \mathcal{B}(X) \) be strongly continuous and \( f \in C([a,b], X) \). Then the function
\[
g : M \to X, \quad (t,s) \mapsto S(t,s)f(s),
\]
is also continuous.

Further, let \( Y \subseteq X \) be a subspace and the map \( [a,t] \to X, \ s \mapsto S(t,s)y \), has the derivative \( \partial_s S(t,s)y \) for each \( t \in (a,b] \) and \( y \in Y \). Let \( f \in C^1([a,b], X) \) take values in \( Y \). Then the map \( [a,t] \ni s \mapsto g(t,s) \) is differentiable in \( X \) with
\[
\partial_s g(t,s) = S(t,s)f'(s) + \partial_s S(t,s)f(s).
\]

Proof. Observe that \( \sup_{(t,s) \in M} \| S(t,s)x \| < \infty \) for every \( x \in X \) by continuity. The number \( c := \sup_{(t,s) \in M} \| S(t,s) \| \) is finite by the uniform boundedness principle. For \( (t,s),(t',s') \in M \) we thus obtain
\[
\| S(t',s')f(s') - S(t,s)f(s) \| \leq c \| f(s') - f(s) \| + \| (S(t',s') - S(t,s))f(s) \|,
\]
where the right hand side of this inequality tends to 0 as \( (t',s') \to (t,s) \).

To show the second assertion, fix \( t \in (a,b) \) and take \( s, s+h \in [a,t] \) for \( h \in \mathbb{R} \setminus \{0\} \). We compute
\[
\frac{1}{h}(S(t,s+h)f(s+h) - S(t,s)f(s)) = S(t,s+h)\frac{1}{h}(f(s+h) - f(s)) + \frac{1}{h}(S(t,s+h) - S(t,s))f(s).
\]
As $h \to 0$, the second claim follows from the first part and our assumptions. □

In the next lecture we want to study further properties of generators. To this aim, we will need several concepts which we now explain. Here we only recall the basic definitions, results and examples; most proofs and more details can be found in Appendices A and B.

**Intermezzo 1: Closed operators and their spectra**

Let $D(A) \subseteq X$ be a linear subspace and $A : D(A) \to X$ be linear. The operator $A$ is called closed if it holds:

If $(x_n)_n$ is any sequence in $D(A)$ such that the limits $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} Ax_n = y$ exist in $X$, then $x \in D(A)$ and $Ax = y$.

Note that any operator $A \in \mathcal{B}(X)$ is closed with $D(A) = X$. In the next example we introduce the prototype of an unbounded closed operator.

**Example 2.10.** Let $X = C([0, 1])$ and $Af := f'$ with $D(A) = C^1([0, 1])$. Take a sequence $(f_n)_n$ in $D(A)$ such that $(f_n)_n$, respectively $(f'_n)_n$, converge to $f$, respectively $g$, in $X$. It is a well known fact that then $f$ belongs to $C^1([0, 1])$ and $f' = g$ (see also Remark 2.11 (f) below), which means that $A$ is closed.

Second, consider $A_0 f := f'$ with $D(A_0) := \{ f \in C^1([0, 1]) | f'(0) = 0 \}$. If $(f_n)_n$ is a sequence in $D(A)$ such that $f_n \to f$ and $f_n' \to g$ in $X$ as $n \to \infty$, then we obtain $f \in C^1([0, 1])$ with $f' = g$ as above. Furthermore, $g(0) = f'(0) = \lim_{n \to \infty} f_n'(0) = 0$. Consequently $g \in D(A_0)$ and $A_0 f = g'$, i.e., $A_0$ is closed. ◊

Next, we define the Riemann integral for vector valued functions. Let $a < b$ be real numbers. A (tagged) partition $Z$ of the interval $[a, b]$ is given by finite sequences $(t_k)_{k=0}^m$ and $(\tau_k)_{k=1}^m$ in $[a, b]$ satisfying $t_k-1 < t_k$ and $\tau_k \in [t_{k-1}, t_k]$ for all $k \in \{1, \ldots, m\}$, where $t_0 = a$ and $t_m = b$. We set $\delta(Z) := \max_{k=1,\ldots,m} (t_k - t_{k-1})$. For a function $g \in C([a, b], X)$ we define the Riemann sum $S(g, Z)$ (of $g$ with respect to $Z$) by

$$S(g, Z) := \sum_{k=1}^m g(\tau_k)(t_k - t_{k-1}) \in X.$$  

As for continuous real valued functions, it can be shown that for every sequence $(Z_n)_n$ of (tagged) partitions with $\lim_{n \to \infty} \delta(Z_n) = 0$ the sequence $(S(g, Z_n))_n$ converges in $X$ and that the limit $J$ does not depend on the choice of such $(Z_n)_n$. In this sense we say that $S(g, Z)$ converges in $X$ to $J$ as $\delta(Z) \to 0$. The Riemann integral $\int_a^b g(t) \, dt$ is now defined as this limit, i.e.,

$$\int_a^b g(t) \, dt := \lim_{\delta(Z) \to 0} S(g, Z).$$

The integral has the usual properties known from the real valued case (with similar proofs) like linearity, additivity and validity of the standard estimate

$$\left\| \int_a^b g(t) \, dt \right\| \leq (b-a)\|g\|_{\infty}.$$  

The same definition and results work for piecewise continuous functions.
For the substitution rule extends to this setting. The simple property (g) is used
in the vector valued case (see part (e)) so that the Riemann integral we need later. We especially emphasize that the fundamental
theorem of calculus is valid also in the vector valued case (see part (e)).

Remark 2.11. Let \( A \) be a linear operator on \( X \). Then the following assertions hold.

(a) The operator \( A \) is closed if and only if the graph of \( A \), i.e., the set
\[
\text{gr}(A) := \{(x, Ax) \mid x \in D(A)\},
\]
is closed in \( X \times X \) (endowed with the norm given by \( \| (x, y) \| = \| x \| + \| y \| \)) if
and only if \( D(A) \) is a Banach space with respect to the graph norm \( \| x \|_A := \| x \| + \| Ax \| \). We write \( [D(A)] \) for \( (D(A), \| \cdot \|_A) \).

(b) If \( A \) is closed with \( D(A) = X \), then \( A \) is even continuous (“closed graph theorem”).

(c) Let \( A \) be injective and set \( D(A^{-1}) = R(A) := \{ Ax \mid x \in D(A) \} \). Then \( A \)
is closed if and only if \( A^{-1} \) is closed.

(d) Let \( A \) be closed and \( f \in C([a, b], X) \) with \( f(t) \in D(A) \) for each \( t \in [a, b] \)
such that \( Af \in C([a, b], X) \), where \( (Af)(t) := Af(t) \). We then have
\[
\int_a^b f(t) \, dt \in D(A) \quad \text{and} \quad A \int_a^b f(t) \, dt = \int_a^b Af(t) \, dt.
\]
An analogous result holds for piecewise continuous functions.

(e) For \( f \in C([a, b], X) \) the function
\[
[a, b] \rightarrow X, \quad t \mapsto \int_a^t f(\tau) \, d\tau,
\]
is differentiable with
\[
\frac{d}{dt} \int_a^t f(\tau) \, d\tau = f(t) \quad \text{for all} \ t \in [a, b]. \tag{2.6}
\]
For \( g \in C^1([a, b], X) \) and \( t \in [a, b] \) we have
\[
\int_a^t g'(\tau) \, d\tau = g(t) - g(a). \tag{2.7}
\]

(f) Let \( (f_n)_n \) be a sequence in \( C^1(J, X) \), \( a \in J \), and \( f, g \in C(J, X) \) for an
interval \( J \) such that \( f_n \rightarrow f \) and \( f'_n \rightarrow g \) uniformly on \( J \) as \( n \rightarrow \infty \). Then
\( f \in C^1(J, X) \) and \( f' = g \).

(g) Let \( f \in C([a, b], X) \) and \( t \in [a, b] \). Then \( \frac{1}{h} \int_t^{t+h} f(s) \, ds \rightarrow f(t) \) as \( h \rightarrow 0^+ \).

Proof. Parts (a) and (c) are proved in Lemma A.6 of the appendix. Part (b) can be found in Theorem A.7.

(d) Let \( f \) be as in the statement. Clearly, \( S(f, Z) \in D(A) \) for any partition
\( Z \) of \( [a, b] \) and
\[
AS(f, Z) = \sum_{k=1}^{m} (Af)(\tau_k)(t_k - t_{k-1}) = S(Af, Z) \rightarrow \int_a^b Af(t) \, dt
\]
as \( \delta(Z) \rightarrow 0 \) because \( Af \) is continuous. The assertion now follows from the
closedness of \( A \).
To show (2.7), set \( \varphi(t) = \int_a^t g'(\tau) \, d\tau \) for \( t \in [a, b] \). Equation (2.6) implies that \( \varphi \in C^1([a, b], X) \) with \( \varphi' = g' \). Therefore, \( \varphi - g \) belongs to \( C^1([a, b], X) \) with vanishing derivative. In the proof of Proposition 2.8 we have seen that thus \( \varphi - g \) is constant, and hence (2.7) is true.

(f) Formula (2.7) gives

\[
    f_n(t) = f_n(a) + \int_a^t f'_n(\tau) \, d\tau
\]

for all \( t \in J \). Letting \( n \to 0 \), we deduce that

\[
    f(t) = f(a) + \int_a^t g(\tau) \, d\tau
\]

for all \( t \in J \). Hence, \( f \in C^1(J, X) \) and \( f' = g \) due to (2.6). \( \square \)

It is a delicate matter to add or multiply closed operators. The situation is simpler if one operator is bounded, see Proposition A.9 in Appendix A.

**Remark 2.12.** Let \( A \) be closed and \( T \in B(X) \). Then the operators \( B = A + T \) with \( D(B) = D(A) \) and \( C = AT \) with \( D(C) = \{ x \in X \mid Tx \in D(A) \} \) are closed. This applies in particular to the operator \( \lambda I - A \) for \( \lambda \in \mathbb{C} \).

For a closed operator \( A \), we define the **resolvent set**

\[
    \rho(A) := \{ \lambda \in \mathbb{C} \mid \lambda I - A : D(A) \to X \text{ is bijective} \}.
\]

We write \( R(\lambda, A) \) for \( (\lambda I - A)^{-1} \) if \( \lambda \in \rho(A) \). This operator is called **resolvent**. The **spectrum of** \( A \) is given by \( \sigma(A) := \mathbb{C} \setminus \rho(A) \). Since \( \lambda I - A \) is closed, \( R(\lambda, A) \) is closed with domain \( X \) and thus bounded thanks to the closed graph theorem (see Remark 2.11(b)). It is known that \( \rho(A) \) is open in \( \mathbb{C} \) (and so \( \sigma(A) \) is closed). More precisely, for \( \lambda \in \rho(A) \) we have

\[
    B(\lambda, \| R(\lambda, A) \|^{-1}) \subseteq \rho(A), \tag{2.8}
\]

as one can see by a Neumann series. Moreover, if \( T \in B(X) \), then \( \sigma(T) \) is even compact and always non-empty, and the **spectral radius** of \( T \) is given by

\[
    r(T) := \max \{|\lambda| \mid \lambda \in \sigma(T)\} = \inf_{n \in \mathbb{N}} \| T^n \|^{\frac{1}{n}} = \lim_{n \to \infty} \| T^n \|^{\frac{1}{n}}.
\]

There are closed operators \( A \) with \( \sigma(A) = \mathbb{C} \) or \( \sigma(A) = \emptyset \) (see Example B.3 (b) and (c)). We have the **resolvent equation**

\[
    R(\mu, A) - R(\lambda, A) = (\lambda - \mu)R(\lambda - A)R(\mu, A) = (\lambda - \mu)R(\mu, A)R(\lambda, A)
\]

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for all \( \lambda, \mu \in \rho(A) \). Furthermore, the map \( \rho(A) \rightarrow \mathcal{B}(X), \lambda \mapsto R(\lambda, A) \), is infinitely often differentiable (even analytic) with
\[
\left( \frac{d}{d\lambda} \right)^n R(\lambda, A) = (-1)^n n! R(\lambda, A)^{n+1}
\] (2.9)
for all \( \lambda \in \rho(A) \) and \( n \in \mathbb{N}_0 \). These results are shown in Theorems B.4 and B.6 of the appendix.

**Exercises**

**Exercise 2.1.** Let \( A \in \mathcal{B}(X) \) and \( t \in \mathbb{R} \). Show that the series
\[
e^{tA} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k
\]
converges absolutely in \( \mathcal{B}(X) \) uniformly for \( t \in [-r, r] \), for any \( r > 0 \). Further show that \( \frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A \) for all \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \).

Let \( A, B \in \mathcal{B}(X) \) with \( AB = BA \). Show that \( e^{A+B} = e^A e^B = e^B e^A \). In particular \( e^{(t+s)A} = e^{tA} e^{sA} \) for \( t, s \in \mathbb{R} \) and \( e^{\lambda A} = e^{\lambda A} \) for \( \lambda \in \mathbb{C} \).

**Exercise 2.2.** Let \( p \in [1, \infty) \) and \( X = L^p(\mathbb{R}) \). Set \( T(t)f = f(\cdot + t) \) for \( t \in \mathbb{R} \) and \( f \in L^p(\mathbb{R}) \). Show that \( (T(t))_{t \in \mathbb{R}} \) is a \( C_0 \)-group of isometries on \( X \) and that the map \( \mathbb{R} \rightarrow \mathcal{B}(X), t \mapsto T(t) \), is not continuous.

**Exercise 2.3.** Let \( p \in [1, \infty) \) and \( X = L^p(0, 1) \). For \( t \geq 0 \), \( s \in (0, 1) \) and \( f \in L^p(0, 1) \) set
\[
(T(t)f)(s) := \begin{cases} f(s+t), & s + t < 1, \\ 0, & s + t \geq 1. \end{cases}
\]
Show that \( (T(t))_{t \geq 0} \) is a \( C_0 \)-semigroup on \( X \).

**Exercise 2.4.** Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^d \) be open, \( X = C_0(\Omega) \) and \( m \in C(\Omega) \) such that \( \sup_{s \in \Omega} \text{Re}(m(s)) < \infty \). Define \( T(t)f = e^{tm} f \) for \( t \geq 0 \) and \( f \in X \). Show that \( T(\cdot) \) is a \( C_0 \)-semigroup on \( X \) generated by the operator
\[
Af = mf \quad \text{with} \quad D(A) = \{ f \in X \mid mf \in X \}.
\]

**Exercise 2.5.** Let \( A \) be a closed operator, \( a \in \mathbb{C} \setminus \{0\} \) and \( b \in \mathbb{C} \). Define \( B = aA + b \) with \( D(B) = D(A) \). Show that \( \sigma(B) = a\sigma(A) + b \) and \( R(\mu, A) = \frac{1}{a} R\left( \frac{\mu-b}{a}, A \right) \) for \( \mu \in \rho(B) \).
Bibliography


