The nonlinear wave equation with cubic forcing term

This lecture is devoted to the investigation of the nonlinear wave equation with Dirichlet boundary conditions and a cubic forcing term on a bounded domain in $\mathbb{R}^3$. We first show that the theory of the previous lecture can be applied to this problem and thus derive local wellposedness and regularity of mild solutions. We then focus on the qualitative behavior of the solutions. Depending on the sign of the forcing term, we obtain either global existence or blow-up. As another important feature we establish the finite speed of propagation of the solutions to the wave equation. Finally, we construct a so-called “standing wave solution” of the nonlinear wave equation.

We start with the (real) differentiability of the nonlinearities $F$ arising in this and later lectures. As often in partial differential equations, our applications lead to superposition operators of the form $F(u) = \phi(u)$ for a real differentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}$, where we have $F(u) = iu|u|^\alpha - 1$ for the nonlinear Schrödinger equation. These operators act on $L^p$-spaces of complex-valued functions. However, when differentiating $F$ it is convenient to identify $\mathbb{C}$ with $\mathbb{R}^2$ and to consider $\phi$ as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.

To this aim, we introduce the following notations. Let $z \in \mathbb{C}$. For $\phi : \mathbb{R}^2 \to \mathbb{R}$ and $\phi = (\phi_1, \phi_2) : \mathbb{R}^2 \to \mathbb{R}^2$, we define

$$\varphi(z) = \varphi(\Re z, \Im z) \in \mathbb{R} \quad \text{and} \quad \phi(z) = \phi_1(z) + i\phi_2(z) \in \mathbb{C}.$$  

Moreover, for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $M = (\xi) \in \mathbb{R}^{2 \times 2}$, we set

$$\xi \cdot z = \xi_1 \Re z + \xi_2 \Im z \in \mathbb{R} \quad \text{and} \quad Mz = \xi \cdot z + i\eta \cdot z \in \mathbb{C}.$$  

Our first lemma will allow us to treat the part of the “energy” arising from the nonlinearity in our applications, whereas the second lemma is concerned with the nonlinearity itself.

**Lemma 9.1.** Let $\emptyset \neq U \subseteq \mathbb{R}^d$ be open. Let $\varphi \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfy $|\varphi(z)| \leq c_0|z|^{1+\alpha}$ and $|\nabla \varphi(z)| \leq c_0|z|^\alpha$ for all $z \in \mathbb{C}$ and some constants $c_0 > 0$ and $\alpha \geq 1$. Then the map

$$\Phi : L^{1+\alpha}(U) \to \mathbb{R}, \quad \Phi(u) = \int_U \varphi(u) \, dx,$$

is (real) continuously differentiable. Its derivative $\Phi'(u) \in \mathcal{B}_R(L^{1+\alpha}(U), \mathbb{C})$ at $u \in L^{1+\alpha}(U)$ is given by

$$\Phi'(u)v = \int_U \nabla \varphi(u) \cdot v \, dx, \quad v \in L^{1+\alpha}(U). \quad (9.1)$$

Moreover, $\|\Phi'(u)\|_{\mathcal{B}(L^{1+\alpha}(U), \mathbb{C})} \leq c_0\|u\|_{1+\alpha}^\alpha$ for all $u \in L^{1+\alpha}(U)$ and thus $\Phi'$ is bounded on bounded sets.
We further have \[
\left| \int_U \nabla \varphi(u) \cdot v \, dx \right| \leq \|\nabla \varphi(u)\|_p \|v\|_{1+\alpha} \leq c_0 \|u\|_1^{\alpha} \|v\|_{1+\alpha}, \tag{9.2}
\] due to Hölder’s inequality and the growth assumption on \(\nabla \varphi\). To check the asserted differentiability of \(\Phi\) at \(u\), we compute
\[
\varphi(u(x) + v(x)) - \varphi(u(x)) - \nabla \varphi(u(x)) \cdot v(x)
\]
\[
= \int_0^1 \frac{d}{d\tau} \varphi(u(x) + \tau v(x)) \, d\tau - \nabla \varphi(u(x)) \cdot v(x)
\]
\[
= \int_0^1 (\nabla \varphi(u(x) + \tau v(x)) - \nabla \varphi(u(x))) \cdot v(x) \, d\tau
\]
for a.e. \(x \in U\). Integrating over \(U\) and using Fubini’s theorem, we infer
\[
D_v := \Phi(u + v) - \Phi(u) - \int_U \nabla \varphi(u) \cdot v \, dx = \int_0^1 \int_U (\nabla \varphi(u + \tau v) - \nabla \varphi(u)) \cdot v \, dx \, d\tau.
\]
As above, Hölder’s inequality then implies that
\[
|D_v| \leq \|v\|_{1+\alpha} \int_0^1 \|\nabla \varphi(u + \tau v) - \nabla \varphi(u)\|_p \, d\tau =: \|v\|_{1+\alpha} I(v).
\]
We claim that \(I(v) \to 0\) as \(v \to 0\) in \(L^{1+\alpha}(U)\), which means that \(\Phi\) is (real) differentiable at \(u\) and (9.1) holds. Moreover, the asserted estimate for \(\Phi'\) then follows from (9.2). The claim holds if for each null sequence \((v_n)_n\) in \(L^{1+\alpha}(U)\) there is a subsequence \((v_{n_j})_j\) such that \(I(v_{n_j}) \to 0\) as \(j \to \infty\).

So let \(v_n \in L^{1+\alpha}(U)\) converge to \(0\) in \(L^{1+\alpha}(U)\) as \(n \to \infty\). By (the proof of) the Riesz-Fischer theorem there are a subsequence \((v_{n_j})_j\) and a function \(g \in L^{1+\alpha}(U)\) such that \(v_{n_j} \to 0\) a.e. as \(j \to \infty\) and \(|v_{n_j}| \leq g\) a.e. for all \(j \in \mathbb{N}\). For \(\tau \in [0, 1]\), the growth assumption on \(\nabla \varphi\) thus implies the pointwise estimate
\[
|\nabla \varphi(u + \tau v_{n_j}) - \nabla \varphi(u)| \leq c_0 (|u| + |g|)^{\alpha} + c_0 |u|^{\alpha} =: f \in L^p(U).
\]
The theorem of dominated convergence then shows that the integrand of \(I(v_{n_j})\) tends to \(0\) as \(j \to \infty\) for each \(\tau \in [0, 1]\). Since this integrand is bounded by the constant \(\|f\|_p\), we further derive that \(I(v_{n_j}) \to 0\) as \(j \to \infty\), as asserted.

In the same way one proves that \(\Phi'(w) \to \Phi'(u)\) in \(B_\mathbb{R}(L^{1+\alpha}(U), \mathbb{C})\) as \(w \to u\) in \(L^{1+\alpha}(U)\), i.e., \(\Phi'\) is continuous. \(\square\)

**Lemma 9.2.** Let \(\phi = (\phi_1, \phi_2) \in C^1(\mathbb{R}^2, \mathbb{R}^2)\) satisfy \(|\phi(z)| \leq c_0 |z|^{\alpha}\) and \(|\phi'(z)| \leq c_0 |z|^{\alpha-1}\) for all \(z \in \mathbb{C}\) and constants \(c_0 > 0\) and \(\alpha > 1\), where \(\phi'(z) = \phi'(\text{Re } z, \text{Im } z)\). Let \(p \in [\alpha, \infty)\) and \(\emptyset \neq U \subseteq \mathbb{R}^d\) be open. Then the map
\[
F : L^p(U) \to L^p(U), \quad F(u) = \phi(u) = \phi_1(u) + i\phi_2(u),
\]
is (real) continuously differentiable and its derivative at \(u \in L^p(U)\) is given by
\[
F'(u)v = \phi'(u)v = \nabla \phi_1(u) \cdot v + i \nabla \phi_2(u) \cdot v, \quad v \in L^p(U).
\]
We further have \(\|F'(u)\|_{B_\mathbb{R}(L^p, L^{p(\alpha)})} \leq c_0 \|u\|_{p(\alpha)}^{\alpha-1}\), so that the derivative is bounded on bounded sets.
PROOF. Let \( u, v \in L^p(U) \). As in the previous lemma, the growth assumptions imply that \( F(u) \in L^\frac{p}{q}(U) \) and that \( |\phi'(u)| \in L^q(U) \) with \( q = \frac{p}{\alpha - 1} \). Hölder’s inequality with exponents \( \frac{q}{p} = \frac{1}{p} + \frac{\alpha - 1}{p} \) yields that \( v \mapsto \phi'(u)v \) belongs to \( B_{\mathbb{R}}(L^p(U)), L^\frac{p}{q}(U) \)). We further obtain the pointwise identity

\[
F(u + v) - F(u) = \int_0^1 (\phi'(u + \tau v) - \phi'(u))v \, d\tau.
\]

Minkowski’s inequality for integrals and Hölder’s inequality then imply

\[
\|F(u + v) - F(u) - \phi'(u)v\| \leq \int_0^1 \|\phi'(u + \tau v) - \phi'(u))v\| \, d\tau
\]

\[
\leq \|v\|_p \int_0^1 \|\phi'(u + \tau v) - \phi'(u))\| \, d\tau.
\]

The same arguments as in the previous proof show that the above integral converges to zero as \( v \to 0 \) in \( L^p(U) \), where one has to use the growth conditions on \( \phi' \). Therefore \( F : L^p(U) \to L^\frac{p}{q}(U) \) is (real) differentiable and its derivative \( F'(u) \) can be represented as asserted. The continuity of \( u \mapsto F'(u) \) and the norm bound for \( F'(u) \) follow as before. \( \square \)

We now apply the above lemmas to the nonlinear maps used below. To use complex notation, for \( z, w \in \mathbb{C} \) we set \( z \cdot w = \text{Re} z \text{Re} w + \text{Im} z \text{Im} w \in \mathbb{R} \) and note that \( \text{Re}(zw) = z \cdot w \).

**Corollary 9.3.** Let \( \alpha > 1, \beta \geq 1, p \in [\alpha, \infty) \) and let \( U \subseteq \mathbb{R}^d \) be open. Then the maps

\[
\Phi : L^{1+\beta}(U) \to \mathbb{R}, \quad \Phi(u) = \frac{1}{1+\beta} \int_U |u|^{1+\beta} \, dx,
\]

\[
F : L^p(U) \to L^\frac{p}{\alpha-1}(U), \quad F(u) = |u|^\alpha - 1 u,
\]

are (real) continuously differentiable, Lipschitz on bounded sets and their derivatives are given by

\[
\Phi'(u)v = \int_U |u|^{\beta-1} \text{Re}(u \overline{v}) \, dx \quad \text{for } u, v \in L^{1+\beta}(U),
\]

\[
F'(u)v = |u|^{\alpha-1}v + (\alpha - 1)|u|^{\alpha-3}u \text{Re}(u \overline{v}) \quad \text{for } u, v \in L^p(U).
\]

**Proof.** Set \( \varphi(z) = \frac{1}{1+\beta} |z|^{1+\beta} \) and \( \phi(z) = |z|^{\alpha-1}z \) for \( z \in \mathbb{C} \cong \mathbb{R}^2 \). The growth conditions for \( \varphi \) and \( \phi \) from Lemma 9.1 and 9.2 clearly hold. Writing \( r = \text{Re} z \) and \( s = \text{Im} z \) for \( z \in \mathbb{C} \) and identifying \( z \) and \( (r, s) \), we obtain

\[
\nabla \varphi(z) = \frac{1}{1+\beta} \left( \partial_r (r^2 + s^2)^{\frac{\beta+1}{2}}, \partial_s (r^2 + s^2)^{\frac{\beta+1}{2}} \right)^\top = |z|^{\beta-1}z,
\]

\[
\phi'(z) = (\partial_r (r^2 + s^2)^{\frac{\alpha-1}{2}} (z), \partial_s (r^2 + s^2)^{\frac{\alpha-1}{2}} (z))
\]

\[
= |z|^{\alpha-3} \begin{pmatrix} (\alpha - 1)r^2 + |z|^2 & (\alpha - 1)rs \\ (\alpha - 1)rs & (\alpha - 1)s^2 + |z|^2 \end{pmatrix}
\]

for \( z \neq 0 \), and \( \phi'(0) = 0 \). As a result, also \( \nabla \varphi \) and \( \phi' \) satisfy the growth assumptions of the two previous lemmas. Moreover, for \( w \in \mathbb{C} \) with \( \rho = \text{Re} w \) and \( \sigma = \text{Im} w \),
we compute \( \nabla \varphi(z) \cdot w = |z|^\beta - 1 \Re(z\overline{w}), \ (r, s)(r\rho + s\sigma) = z \Re(z\overline{w}) \) and
\[
\phi'(z)w = |z|^{\alpha-1}w + (\alpha - 1)|z|^{\alpha-3}z \Re(z\overline{w}), \quad z \neq 0.
\]
The assertions now follow from Lemmas 9.1 and 9.2.

We can now treat the nonlinear wave equation (1.4) with Dirichlet boundary conditions on a bounded open set \( \emptyset \neq U \subseteq \mathbb{R}^3 \). In the same way as for the linear wave equation (6.6) we rewrite (1.4) as an evolution equation in \( L^2(U) \) having second order in time, where we use again the Dirichlet Laplacian from Example 5.12. We thus obtain the problem
\[
w''(t) = \Delta_D w(t) - aw(t)|w(t)|^2, \quad t \in J,
\]
\[
w(0) = w_0, \quad w'(0) = w_1,
\]
where \( w_0 \in D(\Delta_D), w_1 \in \dot{H}^1(U) \) and \( a \in \mathbb{R} \) are given. We look for solutions
\[w \in C^2(J, L^2(U)) \cap C^1(\dot{H}^1(U)) \cap C(J, [D(\Delta_D)])\]
of (9.3). To solve (9.3), we proceed as in Lecture 6 and reformulate (9.3) as the semilinear evolution equation
\[
u'(t) = Au(t) + F(u(t)), \quad t \in J, \quad u(0) = u_0,
\]
on the Hilbert space \( X = \dot{H}^1(U) \times L^2(U) \) endowed with the norm given by
\[\|(u_1, u_2)\|^2 = \|\nabla u_1\|^2 + \|u_2\|^2.\]
Here we set \( u_0 = (u_0, w_1) \), and
\[A = \begin{pmatrix}
0 & I \\
\Delta_D & 0
\end{pmatrix} \quad \text{with} \quad D(A) = D(\Delta_D) \times \dot{H}^1(U),
\]
\[F(u) = (0, -au_1|u_1|^2) = (0, F_0(u_1)) \quad \text{for} \quad u = (u_1, u_2) \in X.
\]

Recall that \( A \) is skewadjoint by Example 5.13. Moreover, Corollary 9.3 implies that \( F_0 : L^3(U) \rightarrow L^2(U) \) is (real) continuously differentiable and Lipschitz on bounded sets. Since \( \dot{H}^1(U) \rightarrow L^3(U) \) by Sobolev’s embedding (5.7), we obtain that \( F : X \rightarrow X \) has the same properties. In particular, \( F \) satisfies (8.1). We point out that this reasoning crucially depends on the fact that \( U \subseteq \mathbb{R}^3 \) and that the nonlinearity is cubic (i.e., \( \alpha = 3 \) in Corollary 9.3).

We can now apply Theorems 8.6 and 8.10 to (9.4) with \( A \) and \( F \) as above. Exactly as in Lemma 6.10, one shows that solutions of (9.3) and (9.4) are in unique correspondence, where \( u = (w, w') \).

To define mild solutions for (9.3), we note that we can write the unitary group \( T(\cdot) \) generated by \( A \) as
\[
T(t) = \begin{pmatrix}
T_{11}(t) & T_{12}(t) \\
T_{21}(t) & T_{22}(t)
\end{pmatrix}
\]
for operators \( T_{11}(t) \in \mathcal{B}(\dot{H}^1(U)), T_{12}(t) \in \mathcal{B}(L^2(U), \dot{H}^1(U)), T_{21}(t) \in \mathcal{B}(\dot{H}^1(U), L^2(U)) \) and \( T_{22}(t) \in \mathcal{B}(L^2(U)) \), where \( t \in \mathbb{R} \). Duhamel’s formula (8.3) thus leads to the integral equation
\[
w(t) = T_{11}(t)w_0 + T_{12}(t)w_1 + \int_0^t T_{12}(t-s)F_0(w(s)) \, ds, \quad t \in J,
\]
on \( \dot{H}^1(U) \). We call a function \( w \in C(J, \dot{H}^1(U)) \) satisfying (9.6) a mild solution of (9.3), where \( w_0 \in \dot{H}^1(U) \) and \( w_1 \in L^2(U) \) are given.

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After these preparations we can now establish a local wellposedness result for the nonlinear wave equation (9.3).

**Proposition 9.4.** Let $U \subseteq \mathbb{R}^3$ be open and bounded, and let $a \in \mathbb{R}$. Then for each $u_0 = (w_0, w_1) \in X = H^1(U) \times L^2(U)$ there is a unique mild solution $w \in C([0, t^+(u_0)], \dot{H}^1(U))$ of (9.3). Moreover, let $J^+ = [0, t^+(u_0))$. Then the following assertions hold.

(a) A mild solution $w$ of (9.3) belongs to $C^1(J^+, L^2(U))$ and $u = (w, w')$ is the mild solution of (9.4).

(b) If $u_0$ and $w_1$ are real-valued, then $w$ is real-valued.

(c) If $(w_0, w_1) \in D(A) = D(\Delta_D) \times \dot{H}^1(U)$, then the corresponding mild solution $w$ belongs to $C^2(J^+, L^2(U)) \cap C^1(J^+, \dot{H}^1(U)) \cap C(J^+, [D(\Delta_D)])$ and it solves (9.3).

(d) If $t^+(u_0) < \infty$, then $\lim_{t \to t^+(u_0)} (\|\nabla w(t)\|^2 + \|w'(t)\|^2) = \infty$.

(e) For any given $b \in (0, t^+(u_0))$ there is a radius $r > 0$ such that for $\tilde{u}_0 = (\tilde{w}_0, \tilde{w}_1) \in B_X(u_0, r)$, we have $t^+(\tilde{u}_0) > b$ and the map $B_X(u_0, r) \to C([0, b], X), \tilde{u}_0 \to (\tilde{w}, \tilde{w}')$ is Lipschitz. Here, $\tilde{w}$ is the mild solution of (9.3) with the initial values $(\tilde{w}_0, \tilde{w}_1)$.

**Proof.** 1) We first note that due to the above observations, Theorem 8.10 yields a unique solution $u$ of (9.4) on $J^+ = [0, t^+(u_0))$ for $A$ and $F$ given by (9.5) if $u_0 = (w_0, w_1) \in D(A)$. We have $u = (w, w')$, where $w \in C^2(J^+, L^2(U)) \cap C^1(J^+, \dot{H}^1(U)) \cap C(J^+, [D(\Delta_D)])$ solves (9.3), so that (c) holds.

To obtain a mild solution for (9.3), let $u_0 = (w_0, w_1) \in X$. We then find $u_n = (w_n, w_{1n}) \in D(A)$ converging to $u_0$ in $X$ as $n \to \infty$. Theorem 8.6 shows that the corresponding solutions $u_n = (w_n, w_{1n})$ tend in $C([0, b], X)$ to the mild solution $u$ of (9.4) for the initial value $u_0$, as $n \to \infty$, where $b \in (0, t^+(u_0))$ is arbitrary. By definition, the first component $w = [u]_1$ is a mild solution of (9.3). Since $w_n$ and $w_n'$ converge in $C([0, b], L^2(U))$, the function $w$ belongs to $C^1([0, b], L^2(U))$ and $u = (w, w')$. This also proves (a).

2) We next show that mild solutions of (9.3) are unique. Let $w, \tilde{w} \in C([0, b], \dot{H}^1(U)) =: E$ be mild solutions of (9.3) for some $b > 0$ with the same initial values $(w_0, w_1) \in X$. Let $r$ be larger than the norms of $w$ and $\tilde{w}$ in $E$. Since $\|T(t)\| = 1$, it is easy to see that $\|T_{12}(t)\|_{B(L^2, \dot{H}^1)} \leq 1$ for all $t \in \mathbb{R}$. From (9.6), (8.1) and (5.7), we then deduce

$$\|w(t) - \tilde{w}(t)\|_{1,2} = \left\| \int_0^t T_{12}(t-s)(F_0(w(s)) - F_0(\tilde{w}(s)))\,ds \right\|_{1,2} \leq \int_0^t \|F_0(w(s)) - F_0(\tilde{w}(s))\|_2\,ds \leq cL(r)\int_0^t \|w(s) - \tilde{w}(s)\|_{1,2}\,ds$$

for all $t \in [0, b]$. Gronwall’s inequality thus implies that $w = \tilde{w}$.

3) The assertions (d) and (e) now directly follow from (a) and Theorem 8.6. To show (b), let $w_0$ and $w_1$ be real-valued. Due to part (e), we may assume that $(w_0, w_1) \in D(A)$. We set $g = \text{Re} \, w$ and $v = \text{Im} \, w$. Then $v \in C^2(J^+, L^2(U)) \cap C^1(J^+, \dot{H}^1(U)) \cap C(J^+, [D(\Delta_D)])$ solves

$$v''(t) = \Delta_D v(t) - ag(t)^2 v(t) - av(t)^3, \quad t \in J^+, \quad v(0) = 0, \quad v'(0) = 0,$$
where \( g \) is considered as a given function. Since \( g(t) \) is uniformly bounded in \( \dot{H}^1(U) \cong L^6(U) \) for \( t \in [0, b] \) and any \( b \in J^+ \), Hölder’s inequality yields
\[
\|g(t)^2 v(t)\|_2 \leq \|g(t)\|_6^2 \|v(t)\|_6 \leq c(b)\|v(t)\|_6^2 \text{ for } t \in [0, b] \text{ and a constant possibly depending on } b.
\]
For \( t \in [0, b] \) one can now estimate as in step 2),
\[
\|v(t)\|_{1, 2} \leq \int_0^t \left(\|ag(s)^2 v(s)\|_2 + \|av(t)^3\|_2\right) ds \leq |a|(c(b) + 1)c \int_0^t \|v(s)\|_{1, 2} ds,
\]
using once more Sobolev’s embedding (5.7). Gronwall’s inequality then shows that \( v = 0 \), and hence \( w \) is real-valued.

Let \( J^+ = [0, t^+(w_0, w_1)) \) for \( (w_0, w_1) \in X \). Since the mild solution \( w \) of (9.3) belongs to \( C(J^+, \dot{H}^1(U)) \cap C^1(J^+, L^2(U)) \), we can define its “energy”
\[
E_w(t) = E(w(t), w'(t)) = \int_U \left( \frac{1}{2}\|w'(t)\|^2_X + \frac{1}{4}\|\nabla w(t)\|^2 + \frac{a}{4}\|w(t)\|^4 \right) dx
\]
for \( t \in J^+ \). In fact, the map \( X \to \mathbb{R}, (w_0, w_1) \mapsto E(w_0, w_1) \), is well-defined and continuous since \( \dot{H}^1(U) \to L^4(U) \) by Sobolev’s embedding (5.7).

We next show that \( E \) is constant along mild solutions of (9.3) so that it is a natural quantity for the nonlinear wave equation. Further observe that \( E_w(t)^{\frac{1}{4}} \) controls the \( \dot{H}^1(U) \times L^2(U) \) norm of \((w(t), w'(t))\), provided \( a \geq 0 \). This fact leads to global existence of all mild solutions in this case.

**Proposition 9.5.** Let \( U \subseteq \mathbb{R}^3 \) be open and bounded, \( a \in \mathbb{R} \), and \((w_0, w_1) \in X \). The following assertions hold for each mild solution \( w \) of (9.3).

(a) \( E_w(t) = E_w(0) = \frac{1}{2}\|\omega_0\|^2_X + \frac{a}{4}\|w_0\|^4 \) for \( t \in [0, t^+(w_0, w_1)) \).

(b) If \( a \geq 0 \), then \( t^+(w_0, w_1) = \infty \) for all initial values \((w_0, w_1) \in X \).

**Proof.** (a) First let \((w_0, w_1) \in D(A)\) and denote by \( w \) the solution of (9.3) on \( J^+ = [0, t^+(w_0)) \). Employing the regularity of \( w \), Corollary 9.3 and the chain rule, we infer that \( E_w \in C^1(J^+) \) and
\[
E'_w(t) = \int_U \text{Re}(w'(t)\overline{w''(t)} + \nabla w(t) \cdot \nabla w'(t) + a|w(t)|^2 w(t)\overline{w'(t)}) \, dx
\]
\[
= \text{Re} \int_U \overline{w'(t)}(w''(t) - \Delta_D w(t) + aw(t)|w(t)|^2) \, dx = 0
\]
for \( t \in J^+ \). Here we used the definition of \( \Delta_D, \text{Re}(z\overline{w}) = \text{Re}(\overline{z}w) \), and equation (9.3). As a result, \( E_w(t) = E_w(0) \) for all \( t \in J^+ \). Since the map \((w_0, w_1) \mapsto E_w(t) \) is continuous from \( X \) to \( \mathbb{R} \) by Proposition 9.4 (e) and \( \dot{H}^1(U) \hookrightarrow L^2(U) \), we obtain (a) by approximation.

(b) If \( a \geq 0 \), then \( \|w(t), w'(t)\|^4_X \leq 2E_w(t) = 2E_w(0) \) for all \( t \in J^+ \) so that the blow-up criterion from Proposition 9.4 (d) implies \( t^+(w_0, w_1) = \infty \). \( \square \)

We want to repeat the main features of the above prototypical proof of global solvability. One first shows that the time derivative of the energy vanishes along (sufficiently smooth) solutions.\(^1\) Here it is crucial that the energy fits to the equation. By approximation, we then deduce that the energy stays constant

\(^1\)Of course, the argument also works if \( E_w' \leq 0 \).
along all mild solutions using the continuous dependence on initial data and the continuity of the map $X \to \mathbb{R}, (w_0, w_1) \mapsto E(w_0, w_1)$, defined in (9.7). If the energy dominates the norm, the energy equality combined with the blow-up criterion in Proposition 9.4 (d) finally yield global existence.

Now, what is happening if $a < 0$? It is useful to neglect spatial derivatives for a moment and to consider the corresponding ordinary differential equation $\phi'' = |a| \phi^3$, with $\phi(t) \in \mathbb{R}$. This equation has the blow-up solutions $\phi_c(t) = c\left(1 - \sqrt{\frac{|a|}{2}}ct\right)^{-1}$ for each $c > 0$. Therefore, if we consider the cubic wave equation on $U$ with Neumann boundary conditions $\partial_\nu w(t) = 0$ instead of Dirichlet conditions on $\partial U$, then we obtain the exploding solutions $w(t) = \phi_c(t)1$, for $0 \leq t < \sqrt{\frac{2}{|a|}c}$.

In the Dirichlet case one can also derive blow-up for certain initial values, but here one has to argue in a different way. We follow Theorem 2.1 in [Gla73]. To this aim, we recall a few facts about the spectrum and the eigenfunctions of $\Delta_D$, which are also used in Proposition 9.8 below.

We first note that $\hat{H}^1(U)$ is compactly embedded into $L^2(U)$. In fact, one can extend each $f \in \hat{H}^1(U)$ by 0 to a function $\tilde{f} \in H^1(V)$ for an open ball $V$ containing $U$. Due to Theorem D.24 in Appendix D, the space $H^1(V)$ is compactly embedded in $L^2(V)$ which implies the claim. (See also Remark 20 in Section 9.4 in [Bre11].) Observe that $[D(\Delta_D)]$ is continuously embedded into $\hat{H}^1(U)$ since for $u \in D(\Delta_D)$ we have $u \in \hat{H}^1(U)$ and

$$\int_U |\nabla u|^2 \, dx = \int_U \Delta_D u \, dx \leq \|\Delta_D u\|_2 \|u\|_2 \leq \frac{1}{2} \|\Delta_D u\|_2^2 + \frac{1}{2} \|u\|_2^2.$$  (9.8)

by the definition of $\Delta_D$ and Hölder’s inequality. In particular, $[D(\Delta_D)]$ is compactly embedded into $L^2(U)$ and thus $\Delta_D$ has a compact resolvent. Therefore $\sigma(\Delta_D)$ only consists of isolated eigenvalues of finite multiplicity, see e.g. Theorem III.6.9 in [Kat95]. Example 5.12 tells us that $\sigma(\Delta_D) \subseteq (-\infty, 0)$. Hence there is a maximal eigenvalue $-\lambda_0 < 0$ of $\Delta_D$. It is further known that there is a strictly positive eigenfunction $\phi_0$ of $\Delta_D$ for the eigenvalue $-\lambda_0$. The kernel of $\lambda_0 I + \Delta_D$ is one dimensional (and hence the range of $\lambda_0 I + \Delta_D$ has codimension 1 by compactness), if $U$ is connected. See e.g. Theorem 6.5.2 in [Eva10].

After these preparations we state the blow-up result for (9.3) with $a < 0$.

**Proposition 9.6.** 2 Let $U \subseteq \mathbb{R}^3$ be open and bounded and let $a < 0$. Then there is a number $R > 0$ such that for all $(w_0, w_1) \in D(A)$ with $w_0, w_1 \geq R$ on $U$ the corresponding solution $w$ of (9.3) has a finite existence time $t^+(w_0, w_1) < \infty$. We even have $\|w(t)\|_2 \to \infty$ as $t \to t^+(w_0, w_1)$.

**Proof.** 1) Let $(w_0, w_1) \in D(A)$ be real-valued and positive, and let $w$ be the corresponding real-valued solution of (9.3). Suppose that $t^+(w_0, w_1) = \infty$. Take an eigenfunction $\varphi_0 > 0$ of $\Delta_D$ for its largest eigenvalue $-\lambda_0 < 0$ with $\int_U \varphi_0 \, dx = 1$. We set $\phi(t) = \int_U \varphi_0 w(t) \, dx$ for $t \geq 0$. We want to obtain a time $t_0 > 0$ such that $\phi(t) \to \infty$ as $t \to t_0^-$ if $w_0, w_1 > 0$ are sufficiently large. Since $\phi(t) \leq \|\varphi_0\|_2 \|w(t)\|_2$, this will contradict the assumption that $t^+(w_0, w_1) = \infty$.

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2*Warning:* There is a corrected version of this proposition which can be downloaded from the webpage of Lecture 9: [https://isem.math.kit.edu/index.php5/Lecture_9](https://isem.math.kit.edu/index.php5/Lecture_9)
Since \( w \) solves (9.3) and \( \Delta_D \varphi_0 = -\lambda_0 \varphi_0 \), we first obtain \( \phi \in C^2(\mathbb{R}_+) \) and

\[
\phi''(t) = \int_U \varphi_0 w''(t) \, dx = \int_U \varphi_0 (\Delta_D w(t) + |a| w(t)^3) \, dx
\]
\[
= \int_U w(t) \Delta_D \varphi_0 \, dx + |a| \int_U w(t)^3 \varphi_0 \, dx
\]
\[
\geq -\lambda_0 \int_U \varphi_0 w(t) \, dx + |a| \left( \int_U w(t) \varphi_0 \, dx \right)^3 = |a| \phi(t)^3 - \lambda_0 \phi(t),
\]
also using Hölder’s inequality with respect to the measure \( \varphi_0 \, dx \). The blow-up of \( \phi \) will follow from this differential inequality by elementary arguments.

\( \lambda \) Observe that \( |a|s^3 - \lambda_0 s \geq \frac{1}{2} |a|s^3 > 0 \) for \( s \geq s_0 := \sqrt{2|a|^{-1}} > 0 \). We choose \( w_0 \geq 0 \) such that \( \phi(0) > s_0 \). Let \( t_1 \in (0, \infty) \) be the supremum of all \( t \geq 0 \) such that \( \phi \geq s_0 \) on \([0, t]\). We have \( \phi'(0) = \int w_1 \varphi_0 \, dx \geq 0 \) because \( w_1 \geq 0 \). Integrating (9.9), we then derive

\[
\phi'(t) \geq \phi'(0) + \int_0^t \frac{|a|}{2} \phi(\tau)^3 \, d\tau > 0
\]
for \( t \in [0, t_1) \). Hence, \( \phi \) strictly increases on \([0, t_1)\) and thus \( t_1 = \infty \). In particular, \( \phi'(t) > 0 \) and (9.9) yields \( \phi''(t) \geq \frac{|a|}{2} \phi(t)^3 \) for all \( t \geq 0 \). It follows that \( \frac{d}{dt} \frac{1}{2} (\phi')^2 = \phi'' \phi' \geq \frac{|a|}{2} \phi^3 \phi' \) on \( \mathbb{R}_+ \). Integrating once more, we arrive at

\[
\phi'(t)^2 \geq \phi'(0)^2 + 2 \int_0^t \frac{|a|}{2} \phi(\tau)^3 \phi'(\tau) \, d\tau = \phi'(0)^2 + \int_{\phi(0)}^{\phi(t)} |a| r^3 \, dr
\]
\[
= \phi'(0)^2 + \frac{|a|}{4} \phi(t)^4 - \frac{|a|}{4} \phi(0)^4.
\]
Choosing \( w_1 \) such that \( \phi'(0)^2 \geq \frac{|a|}{4} \phi(0)^4 \), we derive \( \phi'(t) \geq \frac{1}{2} \sqrt{|a|} \phi(t)^2 \). The comparison principle for ordinary differential equations then implies the desired blow-up of \( \phi \).

An important feature of wave equations is the finite speed of propagation of their solutions. Roughly speaking, this means that if an initial value is compactly supported, then the support of \( w(t) \) moves with finite velocity. This behavior is in accordance with the theory of relativity, in contrast to the diffusion equation \( u' = \Delta_D u \) whose solutions \( u \) are strictly positive for each \( t > 0 \) if \( u(0) = 0 \) is nonzero. To describe this behavior, we consider the cone

\[
C(x_0, t_0) = \left\{ (x, t) \in \mathbb{R}^3 \times \mathbb{R}_+ \mid 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t \right\}
\]
with base \( \mathcal{B}(x_0, t_0) \subseteq \mathbb{R}^3 \) and vertex \( (x_0, t_0) \in \mathbb{R}^4 \), where \( t_0 > 0 \) and \( x_0 \in \mathbb{R}^3 \). Recall the comment after Example 5.12 stating that \( [D(\Delta_D)] \) is isomorphic to \( H^2(U) \cap \tilde{H}^1(U) \) if \( \partial U \in C^2 \) so that \( \Delta_D u = \Delta u \) for all \( u \in D(\Delta_D) \).

**Proposition 9.7.** Let \( U \subseteq \mathbb{R}^3 \) be open and bounded with \( \partial U \in C^2 \) and let \( a \geq 0 \). Take \( x_0 \in U \) and \( t_0 > 0 \) such that \( \mathcal{B}(x_0, t_0) \subseteq U \). Let \( (w_0, w_1) \in D(\Delta_D) \times \tilde{H}^1(U) \) and assume that \( w_0 = 0 \) on \( B(x_0, t_0) \). Then the corresponding solution \( w \) of (9.3) vanishes on \( C(x_0, t_0) \).

**Proof.** The result is proved by a local energy estimate. Let \( t_0 > 0 \) and \( x_0 \in U \) with \( \mathcal{B}(x_0, t_0) \subseteq U \). Take \( (w_0, w_1) \in D(\Delta_D) \times \tilde{H}^1(U) \) and let \( w \) solve
solve the “nonlinear eigenvalue problem” \( \omega \). Because of cf. the proof of Proposition 9.5 We note that the traces on \( e \) for \( \lambda \) theorem, see e.g. Theorem I.5.1 in \( [95] \). To this aim, we set

\[
e(t) = \int_{B_t} (\frac{1}{2}|\partial_t w(t)|^2 + \frac{1}{2} |\nabla w(t)|^2 + \frac{a}{4}|w(t)|^4) \, dx.
\]

We will show that \( e' \leq 0 \) so that \( 0 \leq e(t) \leq e(0) = 0 \) for all \( t \in [0, t_0) \), and hence \( w = 0 \) on \( C(x_0, t_0) \). (Actually, in case \( a = 0 \) one deduces \( w = 0 \) on \( C(x_0, t_0) \) from \( \partial_t w = 0 \) on \( C(x_0, t_0) \) and \( w(0) = 0 \) on \( B(x_0, t_0) \).) Using Reynolds’ transport theorem (see Exercise 9.1), we compute

\[
e'(t) = \text{Re} \int_{B_t} (\partial_t w(t)\partial_t \overline{w}(t) + \nabla w(t) \cdot \nabla \overline{w}(t) + a|w(t)|^2 w(t)\partial_t \overline{w}(t)) \, dx
\]

\[
- \int_{\partial B_t} (\frac{1}{2}|\partial_t w(t)|^2 + \frac{1}{2} |\nabla w(t)|^2 + \frac{a}{4}|w(t)|^4) \, d\sigma,
\]

cf. the proof of Proposition 9.5 We note that the traces on \( \partial B_t \) exists since \( \partial_t w(t), \partial_t w(t) \in H^1(U) \) and \( w(t) \in H^2(U) \rightarrow C(\overline{U}) \) (see Corollary D.21 and Theorem D.23). Because of \( w(t) \in H^2(U) \), we can use Gauß’ formula from Theorem D.28 on the ball \( B_t \) leading to

\[
\int_{B_t} \nabla w(t) \cdot \nabla \partial_t \overline{w}(t) = - \int_{B_t} \Delta w(t) \partial_t \overline{w}(t) \, dx + \int_{\partial B_t} \partial_w w(t) \partial_t \overline{w}(t) \, d\sigma,
\]

where \( \partial_w w = \nu \cdot \text{tr} \nabla w \) for \( w \in H^2(B_t) \) and the outer unit normal \( \nu \) on \( \partial B_t \). Employing equation (9.3) and Hölder’s inequality, we thus obtain

\[
e'(t) = \text{Re} \int_{\partial B_t} \partial_w w(t)\partial_t \overline{w}(t) \, d\sigma - \int_{\partial B_t} (\frac{1}{2}|\partial_t w(t)|^2 + \frac{1}{2} |\nabla w(t)|^2 + \frac{a}{4}|w(t)|^4) \, d\sigma
\]

\[
\leq - \int_{\partial B_t} \frac{a}{4}|w(t)|^4 \, d\sigma \leq 0. \quad \Box
\]

It would be nice to see a wave type solution of our wave equation. We thus construct “standing waves” \( w \) for (9.3), i.e., solutions \( w \) of (9.3) given by

\[
w(t, x) = e^{i\omega t} \varphi(x), \quad t \geq 0, \quad x \in U,
\]

where \( \omega \in \mathbb{R} \) and \( \varphi \in D(\Delta_D) \). Inserting this ansatz into (9.3), we infer that \( w \) in (9.10) solves (9.3) (with \( w_0 = \varphi \) and \( w_1 = i\omega \varphi \)) if \( \omega^2 \geq 0 \) and \( \varphi \in D(\Delta_D) \) solve the “nonlinear eigenvalue problem”

\[
\Delta_D \varphi + \omega^2 \varphi - a \varphi |\varphi|^2 = 0.
\]

This observation leads us to the theory of semilinear elliptic equations which is a bit off the main track of the Internet Seminar. Here we focus on a rather simple result.

We want to solve (9.11) via bifurcation theory. It allows to construct solutions to (9.11) as perturbations of the maximal eigenvalue of the linearization. We only need one of most basic results in this direction, the Crandall-Rabinowitz theorem, see e.g. Theorem I.5.1 in [Kie04]. To this aim, we set

\[
\Phi(\lambda, \varphi) = \Delta_D \varphi + \lambda \varphi - a \varphi |\varphi|^2
\]

for \( \lambda \in \mathbb{R} \) and \( \varphi \in D(\Delta_D) \). Clearly, \( \Phi(\lambda, 0) = 0 \) for all \( \lambda \in \mathbb{R} \). We are looking for \( 0 \neq \varphi_\omega \in D(\Delta_D) \) and \( \omega \in \mathbb{R} \) with \( \Phi(\omega^2, \varphi_\omega) = 0 \), i.e., \( (\omega, \varphi_\omega) \) solve (9.11).
To check the assumptions of the Crandall-Rabinowitz theorem, let \( U \) be connected and let all spaces be real. We recall from above that \( \Delta_D \) has the largest eigenvalue \(-\lambda_0\) with corresponding eigenfunction \( \phi_0 \) and that the kernel of \( \Delta_D + \lambda_0 I \) is one dimensional and its range has codimension 1. Due to Corollary 9.3 and \([D(\Delta_D)] \hookrightarrow L^6(U)\), we obtain \( \Phi \in C^1(\mathbb{R} \times [D(\Delta_D)], L^2(U)) \) and \( L := \partial_2 \Phi(\lambda_0, 0) = \Delta_D + \lambda_0 I \), where \( I \) is the embedding of \([D(\Delta_D)]\) into \( L^2(U) \). One can further check that \( \Phi \in C^2(\mathbb{R} \times [D(\Delta_D)], L^2(U)) \) and \( \Lambda := \partial_{12} \Phi(\lambda_0, 0) = I \). Finally, \( \phi_0 = \Lambda \phi_0 \) is orthogonal to the range of \( L \) since
\[
(\Delta_D \varphi + \lambda_0 \varphi | \varphi_0) = (\varphi | \Delta_D \varphi_0 + \lambda_0 \varphi_0) = 0
\]
for all \( \varphi \in D(\Delta_D) \). We have now verified the assumptions Theorem I.5.1 in [Kie04] which gives us \( 0 \neq \varphi_\omega \in D(\Delta_D) \) and \( \omega \in \mathbb{R} \) such that \( \Phi(\omega^2, \varphi_\omega) = 0 \). More precisely, here \( \omega^2 \) is close to \( \lambda_0 \) and \( \varphi_\omega \) is small in the norm of \([D(\Delta_D)]\).

**Proposition 9.8.** Let \( a \in \mathbb{R} \) and \( U \subseteq \mathbb{R}^3 \) be open, bounded and connected. Then for \( \omega^2 \) close to \( \lambda_0 \) there are real-valued functions \( 0 \neq \varphi_\omega \in D(\Delta_D) \) solving (9.11). Hence, the functions given by \( w(t) = e^{\omega t} \varphi_\omega, \ t \geq 0 \), are standing waves of (9.3).
Exercises

Exercise 9.1. Let $U \subseteq \mathbb{R}^d$ be open. Let $x_0 \in \mathbb{R}^d$ and $t_0 > 0$ be such that $\overline{B}(x_0, t_0) \subseteq U$. Set $J = [0, t_0]$ and $B_t = B(x_0, t_0 - t)$. Prove for $f \in C^1(J, L^1(U)) \cap C(J, W^{1,1}(U))$ and each $t \in J$ the identity
\[
\frac{d}{dt} \int_{B_t} f(x, t) \, dx = \int_{B_t} (\partial_t f)(x, t) \, dx - \int_{\partial B_t} f(x, t) \, d\sigma.
\]
This is a special case of Reynolds’ transport theorem.

Exercise 9.2. Let $A$ generate the $C_0$–semigroup $T(\cdot)$ on $X$ and that $F : X \to X$ is Lipschitz on bounded subsets of $X$. Consider the semilinear problem
\[
u'(t) = Au(t) + F(u(t)), \quad t \geq 0, \quad u(0) = u_0,
\]
and suppose that $t^+(u_0) = \infty$ for all $u_0 \in X$. Define $\Psi : \mathbb{R}_+ \times X \to X$ by $\Psi(t, u_0) = u(t; u_0)$, where $u(\cdot; u_0)$ is the unique mild solution with initial value $u_0$. Prove that $\Psi$ is a nonlinear semiflow, i.e., we have $\Psi \in C(\mathbb{R}_+ \times X, X)$ and for any $u_0 \in X$ and $t_1, t_2 \geq 0$ it holds that
\[\Psi(0, u_0) = u_0, \quad \Psi(t_2 + t_1, u_0) = \Psi(t_2, \Psi(t_1, u_0)).\]

Exercise 9.3. Let $A$ generate the $C_0$–semigroup $T(\cdot)$ on $X$. Let $F, F_n : X \to X$ for $n \in \mathbb{N}$ be Lipschitz on bounded sets and suppose that $F_n \to F$ locally uniformly as $n \to \infty$. Fix $u_0 \in X$. Let $u$ be the unique mild solution of
\[u'(t) = Au(t) + F(u(t)), \quad t \geq 0, \quad u(0) = u_0,
\]
with maximal existence time $t^+(u_0, F)$, and let $u_n$ be the unique mild solution of
\[u'_n(t) = Au_n(t) + F_n(u_n(t)), \quad t \geq 0, \quad u_n(0) = u_0,
\]
with maximal existence time $t^+(u_0, F_n)$. Take any $0 < b < t^+(u_0, F)$. Show that $\limsup_{n \to \infty} t^+(u_0, F_n) > b$ and that $u_n \to u$ uniformly on $[0, b]$ as $n \to \infty$. 

Bibliography


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