Polynomial decay of bounded $C_0$-semigroups on Hilbert spaces

Roland Schnaubelt*

In Lecture 7 of the Internet Seminar we have studied the exponential stability of $C_0$–semigroups $T(\cdot)$ on a Banach space $X$ with generator $A$. We have seen that this property implies that $s(A) < 0$, but that this spectral condition does not imply exponential stability. Moreover, in a Hilbert space $X$, Gearhart’s stability theorem says that $T(\cdot)$ is exponentially stable if and only if $s(A) < 0$ and the resolvent $R(\lambda, A)$ is uniformly bounded for $\Re \lambda \geq 0$.

However, in many applications one encounters polynomial decay on $D(A)$, i.e., there are constants $N, \alpha > 0$ such that

\[ \|T(t)x\| \leq N t^{-1/\alpha} \|x\|_A \]

for all $t \geq 1$ and $x \in D(A)$. In recent years a quite satisfying theory was established in this context for bounded $T(\cdot)$, i.e., $\sup_{t \geq 0} \|T(t)\| < \infty$. We assume this property, and thus $s(A) \geq 0$.

In this setting, the estimate (1) implies $i\mathbb{R} \subset \rho(A)$ and

\[ \|R(i\tau, A)\| \leq C |\tau|^\alpha \]

for all $\tau \in \mathbb{R}$ with $|\tau| \geq 1$, for a constant $C > 0$. A more general version of this implication was shown in [2] by means of basic spectral and semigroup theory known from the internet seminar. We note that $i\mathbb{R} \subset \rho(A)$ and (2) yield that $|\Im \lambda| \geq c |\Re \lambda|^{-1/\alpha}$ for all $\lambda \in \sigma(A)$ with $|\Im \lambda| \geq 1$ and a constant $c > 0$; i.e., the spectrum of $A$ approaches $i\mathbb{R}$ at $\pm i\infty$ at most polynomially.

The implication (2) $\Rightarrow$ (1) is wrong in general, as shown by an example in [3]. But this implication holds if $X$ is a Hilbert space. This fact was established in [3] in a very clever (and astonishingly simple) way using Plancherel’s theorem and standard semigroup theory. One can apply this result to various weakly damped linear wave equations, see [4].

The project is based on the first two sections of [2] and [3] and on Example 3 of [4]. A few auxiliary results are taken from [1]. Without proofs, we will use some of the basic properties of the ‘fractional powers’ $(I - A)^\alpha$ of $A$. (See e.g. the monograph by Engel and Nagel.)

References


*Karlsruhe Institute of Technology, Dept. of Mathematics, schnaubelt@kit.edu