LECTURE 14

The asymptotic behavior in the defocusing case

In the previous lecture we have seen that the defocusing nonlinear Schrödinger equation

$$u'(t) = i\Delta u(t) - i|u(t)|^{\alpha-1}u(t), \quad t \in J, \quad u(0) = u_0,$$

is globally solvable in the subcritical case $1 < \alpha < \frac{d+2}{(d-2)+} = \alpha_c$. So for each $u_0 \in H^1(\mathbb{R}^d)$ there is a unique solution $u \in C^1(\mathbb{R}, H^{-1}(\mathbb{R}^d)) \cap C(\mathbb{R}, H^1(\mathbb{R}^d))$ of (14.1). We next inquire how $u(t)$ behaves as $t$ tends to $\pm\infty$.

It turns out that the problem (14.1) has a similar long-term behavior as the free linear Schrödinger equation $v' = i\Delta v$. From Corollary 10.10 we know that the free Schrödinger group $T(\cdot)$ satisfies

$$\|T(t)v_0\|_r \leq c|t|^{\frac{d}{2} - \frac{2}{r}}\|v_0\|_r$$

for all $r \in (2, \infty]$, $t \neq 0$ and $v_0 \in L^r(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. In our main Theorem 14.4 we show that the solution of (14.1) decays in a similar way as $t \to \infty$ if $u_0 \in H^1(\mathbb{R}^d)$ satisfies $|x|u_0 \in L^2(\mathbb{R}^d)$ and $r > 2$ is not too large.

Recall that for $r = 2$ there is no decay since $\|u(t)\|_2 = \|u_0\|_2$ for all $t \in \mathbb{R}$ by Theorem 13.2. In view of this conservation law in $L^2(\mathbb{R}^d)$ and the decay in $L^r(\mathbb{R}^d)$ for $r > 2$, the solution of (14.1) should spread out in space as time evolves. This behavior is in accordance with the dispersive phenomena we discussed a bit in Lecture 10.

 Actually, for $r = 2$ one can show an even closer relationship between (14.1) and the free linear Schrödinger equation. In scattering theory one constructs $u_{\pm} \in H^1(\mathbb{R}^d)$ such that $u(t) - T(t)u_{\pm} \to 0$ in $H^1(\mathbb{R}^d)$ as $t \to \pm\infty$. We mention a few scattering results at the end of the lecture, where we also talk a bit about the long-term behavior in the focusing case.

The proof of the convergence result in $L^r$ depends on explicit formulas for the first and second derivatives of the quantity $\|\|x|u(t)\|_2^2$ for an $H^1$–solution $u$ of (13.1) whose initial value $u_0 \in H^1(\mathbb{R}^d)$ satisfies $|x|u_0 \in L^2(\mathbb{R}^d)$. We note that these formulas hold in the focusing case, too. They are established in Proposition 14.1. We remark that they are also crucial for the blow-up results stated at the end of the last lecture. The proof of Proposition 14.1 depends on lengthy calculations and delicate approximation arguments. As shown in Corollary 14.3, these formulas imply an expression for the energy of the functions $v(t) = e^{-\frac{|x|^2}{4t}}u(t)$ for $t \neq 0$. In the defocusing case this expression leads to the desired decay estimate when combined with the conservation laws, the Gagliardo-Nirenberg inequality and the Gronwall lemma, see Theorem 14.4.
The results of this lecture were established by J. Ginibre and G. Velo in e.g. [GV79] for more general nonlinearities. Our presentation follows (parts of) Sections 6.5, 7.2 and 7.3 of [Caz03].

Throughout we use the spaces

\[ L^2(\mathbb{R}^d) = \{ \varphi \in L^2(\mathbb{R}^d) \mid |\ell| \varphi \in L^2(\mathbb{R}^d) \} \quad \text{and} \quad H^1(\mathbb{R}^d) = L^2(\mathbb{R}^d) \cap H^1(\mathbb{R}^d) \]

where we set \( \ell(x) = x \) for \( x \in \mathbb{R}^d \). They are Hilbert spaces when endowed with the norms given by

\[
\| \varphi \|_{L^2(\mathbb{R}^d)}^2 = \| |\ell| \varphi \|_2^2 + \| \varphi \|_2^2 \quad \text{and} \quad \| \varphi \|_{H^1(\mathbb{R}^d)}^2 = \| \varphi \|_{L^2(\mathbb{R}^d)}^2 + \| \varphi \|_{H^1(\mathbb{R}^d)}^2,
\]

respectively. Observe that \( H^1(\mathbb{R}^d) \) is not contained in \( L^2(\mathbb{R}^d) \).

In our first proposition we show that for \( u_0 \in H^1_0(\mathbb{R}^d) \) the solution \( u \) of (13.1) is continuous in \( L^2(\mathbb{R}^d) \) and we calculate the derivatives of the function

\[
t \mapsto \phi_u(t) = \| |\ell| u(t) \|_2^2.
\]

**Proposition 14.1.** Let \( u_0 \in H^1_0(\mathbb{R}^d) \), \( \alpha \in (1,\alpha_c) \) and \( \mu \in \{-1,1\} \). Then the unique maximal \( H^1 \)–solution \( u \) of the nonlinear Schrödinger equation (13.1) belongs to \( C(J(u_0), L^2(\mathbb{R}^d)) \). Moreover, the function \( \phi_u \) given by \( \phi_u(t) = \| |\ell| u(t) \|_2^2 \), \( t \in J(u_0) \), is twice continuously differentiable and satisfies

\[
\begin{align*}
\phi_u'(t) &= 4 \text{Im} \int_{\mathbb{R}^d} \overline{u}(t) \ell \cdot \nabla u(t) \, \text{d}x, \quad (14.2) \\
\phi_u''(t) &= 16E(u_0) + (4d - 8 \frac{d-2}{\alpha+1}) \mu \| u(t) \|_{\alpha+1}^2 + \frac{\mu}{\alpha+1} \| u_0 \|_{\alpha+1}^2. \quad (14.3)
\end{align*}
\]

for all \( t \in J(u_0) \) and the initial energy \( E(u_0) = \frac{1}{2} \| \nabla u_0 \|_2^2 + \frac{\mu}{\alpha+1} \| u_0 \|_{\alpha+1}^2 \).

Observe that the expression for \( \phi_u'' \) is rather simple and depends on \( u(t) \) continuously with respect to the \( H^1 \)–norm. If the solution \( u \) is sufficiently smooth and decays rapidly enough, one can deduce the assertions of Proposition 14.1 in a direct but tedious way, using (13.1) and integration by parts. To perform such computations in a rigorous way, we need several approximation arguments. Already the weight \( |\ell| \) leads to integrability problems at infinity. To overcome them, as before we use the function \( \gamma_\varepsilon(x) = e^{-\varepsilon |x|^2} \) for \( x \in \mathbb{R}^d \) and \( \varepsilon > 0 \).

**Proof of Proposition 14.1.** 1) We first show that \( u \in C(J(u_0), L^2(\mathbb{R}^d)) \), \( \phi_u \in C^1(J(u_0)) \) and that (14.2) holds. We consider any time interval \( J = [0,b] \subseteq J(u_0) \). Negative times are treated in the same way. For \( \varepsilon > 0 \) we define

\[
\phi_{u,\varepsilon}(t) = \| \gamma_\varepsilon |\ell| u(t) \|_2^2 = \int_{\mathbb{R}^d} \gamma_\varepsilon^2 |\ell|^2 |u(t)|^2 \, \text{d}x, \quad t \in J.
\]

We first consider \( u_0 \in H^2(\mathbb{R}^d) \) so that \( u \in C^1(J,L^2(\mathbb{R}^d)) \cap C(J,H^2(\mathbb{R}^d)) \) by Theorem 13.1. It is then clear that \( \phi_{u,\varepsilon} \in C^1(J) \). Using (13.1) and integrating...
by parts, we derive
\[
\phi_{u,\varepsilon}(t) = 2 \text{Re} \int_{\mathbb{R}^d} \gamma_\varepsilon^2 |\ell|^2 \varphi(t) u'(t) \, dx = 2 \text{Re} \int_{\mathbb{R}^d} \gamma_\varepsilon^2 |\ell|^2 \varphi(t)(\Delta u(t) - \mu |u(t)|^{\alpha-1} u(t)) \, dx
\]
\[
= -2 \text{Re} \int_{\mathbb{R}^d} \left( \nabla (\gamma_\varepsilon^2 |\ell|^2 \varphi(t)) \cdot \nabla u(t) + \mu \gamma_\varepsilon^2 |\ell|^2 |u(t)|^{\alpha+1} \right) \, dx
\]
\[
= -2 \text{Re} \int_{\mathbb{R}^d} \left( -4 \varepsilon \ell \cdot \nabla u(t) \gamma_\varepsilon^2 |\ell|^2 \varphi(t) + 2 \ell \cdot \nabla u(t) \gamma_\varepsilon^2 |\ell|^2 \varphi(t) + \gamma_\varepsilon^2 |\ell|^2 |\nabla u(t)|^2 \right) \, dx
\]
\[
= 4 \text{Im} \int_{\mathbb{R}^d} \gamma_\varepsilon^2 (1 - 2 \varepsilon |\ell|^2) \varphi(t) \ell \cdot \nabla u(t) \, dx
\]
for \( t \in J \). By integration it follows
\[
\phi_{u,\varepsilon}(t) = \|\gamma_\varepsilon |\ell| u_0\|_2^2 + 4 \text{Im} \int_0^t \int_{\mathbb{R}^d} \gamma_\varepsilon (1 - 2 \varepsilon |\ell|^2) \varphi(u(s)) \ell \cdot \nabla u(s) \, dx \, ds. \quad (14.4)
\]
Consider now \( u_0 \in H^1_\alpha(\mathbb{R}^d) \) and approximate it by \( u_{0,n} \in H^2(\mathbb{R}^d) \) in \( H^1(\mathbb{R}^d) \) with corresponding solutions \( u_n \). Theorem 12.5 shows that \( J(u_0) \subseteq J \) for all sufficiently large \( n \) and that \( u_n \to u \) in \( C(J, H^1(\mathbb{R}^d)) \) as \( n \to \infty \). Hence, (14.4) holds for \( u_0 \in H^1_\alpha(\mathbb{R}^d) \) by approximation. Observe that \( |\gamma_\varepsilon (1 - 2 \varepsilon |\ell|^2)| \leq 1 \) for all \( \varepsilon > 0 \) and \( x \in \mathbb{R}^d \). Due to Hölder’s and Young’s inequalities, (14.4) leads to
\[
\phi_{u,\varepsilon}(t) \leq \|\ell|u_0\|_2^2 + 4 \int_0^t \phi_{u,\varepsilon}(s)^{\frac{1}{2}} \|\nabla u(s)\|_2 \, ds
\]
\[
\leq \|\ell|u_0\|_2^2 + 2 \int_0^t \|\nabla u(s)\|_2^2 \, ds \quad (14.5)
\]
for \( t \in J \). From \( u \in C(J, H^1(\mathbb{R}^d)) \) and Gronwall’s lemma we deduce that
\[
\sup_{t>0} \|\phi_{u,\varepsilon}\|_\infty =: C < \infty. \quad (14.5)
\]
As \( \varepsilon \to 0 \), Fatou’s lemma yields \( \phi_u(t) = \|\ell|u(t)\|_2^2 \leq C \) for all \( t \in J \). By means of the majorants \( |\ell|u(t)|^2 \) and \( 4|\ell|u|\nabla u| \)
we can now let \( \varepsilon \to 0 \) in (14.4) to obtain
\[
\phi_u(t) = \|\ell|u(t)\|_2^2 = \|\ell|u_0\|_2^2 + 4 \text{Im} \int_0^t \int_{\mathbb{R}^d} \pi(s) \ell \cdot \nabla u(s) \, dx \, ds \quad (14.5)
\]
for \( t \in J \) and \( u_0 \in H^1_\alpha(\mathbb{R}^d) \). Using again the second majorant, one sees that the right-hand side of (14.5) depends continuously on \( t \), so that the norms \( \|\ell|u(t)\|_2 \) converge to \( \|\ell|u(t_0)|_2 \) as \( t \to t_0 \) in \( J \). Moreover, the functions \( |\ell|u(t) \)
tend pointwise a.e. on \( \mathbb{R}^d \) to \( |\ell|u(t_0) \) as \( t \to t_0 \). Hence, a result by Riesz implies that the map \( t \mapsto |\ell|u(t) \in L^2(\mathbb{R}^d) \) is continuous, see Lemma 1.32 in [Kal].
We can thus differentiate (14.5) with respect to \( t \in J \) and deduce (14.2). This formula implies the continuity of \( \phi_u \).

2) We still have to show that \( \phi_u \in C^2(J(u_0)) \) and (14.3) holds. As a first step, we consider functions \( v \in C(J, H^1(\mathbb{R}^d)) \cap C^1(J, L^2(\mathbb{R}^d)) \) and define
\[
\psi_{v,\varepsilon}(t) = \text{Im} \int_{\mathbb{R}^d} \gamma_\varepsilon \varphi(t) \ell \cdot \nabla v(t) \, dx
\]
for \( \varepsilon > 0 \) and \( t \in J \), where \( J \subseteq \mathbb{R} \) is a compact interval containing 0. Later, the functions \( \psi_{u,\varepsilon} \) shall approximate \( \phi_u \) as \( \varepsilon \to 0 \). We want to show that
\[
\psi_{v,\varepsilon}(t) = \psi_{v,\varepsilon}(0) - \text{Im} \int_0^t \int_{\mathbb{R}^d} v'(s) \left[ 2 \gamma_\varepsilon \ell \cdot \nabla \varphi(s) + (d \gamma_\varepsilon + \ell \cdot \nabla \gamma_\varepsilon) \varphi(s) \right] \, dx \, ds \quad (14.7)
\]
for $t \in J$ and each fixed $\varepsilon > 0$. We see in Lemma 14.2 below that the space $C^1(J, H^1(\mathbb{R}^d))$ is dense in $C(J, H^1(\mathbb{R}^d)) \cap C^1(J, L^2(\mathbb{R}^d))$ with respect to the norm given by $\max\{\|v\|_{C^1(J, H^1)}, \|v\|_{C^1(J, L^2)}\}$. By continuity, it thus suffices to show (14.7) for $v \in C^1(J, H^1(\mathbb{R}^d))$. For such functions, $\psi_{\nu, \varepsilon}$ is continuously differentiable and

$$
\psi'_{\nu, \varepsilon}(t) = \text{Im} \int_{\mathbb{R}^d} \left( \gamma_{\nu} \nu(t) \cdot \nabla v(t) + \gamma_{\nu} \nu(t) \cdot \nabla v'(t) \right) dx \\
= -\text{Im} \int_{\mathbb{R}^d} v'(t) \left[ \gamma_{\nu} \nu(t) \cdot \nabla v(t) + \text{div}(\nu(t) \gamma_{\nu}) \right] dx \\
= -\text{Im} \int_{\mathbb{R}^d} v'(t) \left[ 2 \gamma_{\nu} \nu(t) \cdot \nabla v(t) + (d \gamma_{\nu} + \ell \cdot \nabla \gamma_{\nu}) \nu(t) \right] dx, 
$$

(14.8)

where we used $\text{Im} \, z = -\text{Im} \, z$ for $z \in \mathbb{C}$ and Gauß' formula (5.4). By integration we arrive at (14.7).

3) Temporarily we assume that $u_0 \in H^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. The solution $u$ then belongs to $C^1(J(u_0), L^2(\mathbb{R}^d)) \cap C(J(u_0), H^2(\mathbb{R}^d))$ by Theorem 13.1. We define $\psi_{\nu, \varepsilon}$ by (14.6). Equation (14.7) yields that $\psi_{\nu, \varepsilon} \in C^1(J(u_0))$ and that $\psi_{\nu, \varepsilon}$ is given by (14.8) with $v$ replaced by $u$. Inserting (13.1) into (14.8) for $u$, we infer

$$
\psi'_{\nu, \varepsilon}(t) = -\text{Re} \int_{\mathbb{R}^d} (\Delta u(t) - \mu |u(t)|^{\alpha-1} u(t)) \left[ 2 \gamma_{\nu} \ell \cdot \nabla \nu(t) + (d \gamma_{\nu} + \ell \cdot \nabla \gamma_{\nu}) \nu(t) \right] dx.
$$

(14.9)

We first consider the term in (14.9) involving $\Delta u$. Integrating by parts twice, we compute

$$
-\text{Re} \int_{\mathbb{R}^d} \Delta u(t) \left[ 2 \gamma_{\nu} \ell \cdot \nabla \nu(t) + (d \gamma_{\nu} + \ell \cdot \nabla \gamma_{\nu}) \nu(t) \right] dx \\
= \text{Re} \int_{\mathbb{R}^d} 2(\nabla u(t) \cdot \nabla \gamma_{\nu}) (\ell \cdot \nabla \nu(t)) dx \\
+ \int_{\mathbb{R}^d} (2 \gamma_{\nu} |\nabla u(t)|^2 + \gamma_{\nu} \ell \cdot \nabla |\nabla u(t)|^2) dx \\
+ \text{Re} \int_{\mathbb{R}^d} (d \nabla u(t) \cdot \nabla \gamma_{\nu} + \nabla u(t) \cdot \nabla \gamma_{\nu} + \nabla u(t) \cdot (D^2 \gamma_{\nu}) \ell) \nu(t) dx \\
+ \int_{\mathbb{R}^d} (d \gamma_{\nu} + \ell \cdot \nabla \gamma_{\nu}) |\nabla u(t)|^2 dx \\
= \text{Re} 2 \int_{\mathbb{R}^d} (\nabla u(t) \cdot \nabla \gamma_{\nu}) (\ell \cdot \nabla \nu(t)) dx + 2 \int_{\mathbb{R}^d} \gamma_{\nu} |\nabla u(t)|^2 dx \quad (14.10) \\
+ (d+1) \text{Re} \int_{\mathbb{R}^d} \nabla u(t) \cdot \nabla \gamma_{\nu} dx + \text{Re} \int_{\mathbb{R}^d} \nabla \nu(t) \nabla u(t) \cdot (D^2 \gamma_{\nu}) \ell dx,
$$

where $D^2 \gamma_{\nu}$ is the Hessian matrix. The other terms in (14.9) can be written as

$$
\text{Re} \int_{\mathbb{R}^d} \mu |u(t)|^{\alpha-1} u(t) \left[ 2 \gamma_{\nu} \ell \cdot \nabla \nu(t) + (d \gamma_{\nu} + \ell \cdot \nabla \gamma_{\nu}) \nu(t) \right] dx \\
= 2 \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha-1} \gamma_{\nu} \text{Re}(\ell \cdot \nabla u(t) \nu(t)) dx + \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} (d \gamma_{\nu} + \ell \cdot \nabla \gamma_{\nu}) dx.
$$

(14.11)
We denote the first summand on the right-hand side of (14.11) by $I$. At first let $\alpha \geq 3$. Integrating by parts in the second line, we transform $I$ into

$$I = \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha-1} \gamma \left( \text{div}(\ell |u(t)|^2) - d|u(t)|^2 \right) dx$$

$$= -\mu(\alpha - 1) \int_{\mathbb{R}^d} |u(t)|^{\alpha-3} |u(t)|^2 \gamma \text{ Re}(\pi(t) \nabla u(t) \cdot \ell) dx$$

$$- \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} (\ell \cdot \nabla \gamma + d \gamma) dx$$

$$= -\frac{\alpha - 1}{2} I - \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} (\ell \cdot \nabla \gamma + d \gamma) dx,$$

$$(1 + \frac{\alpha - 1}{2}) I = -\mu \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} (\ell \cdot \nabla \gamma + d \gamma) dx,$$

$$I = -\frac{2\mu}{\alpha + 1} \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} (\ell \cdot \nabla \gamma + d \gamma) dx. \quad (14.12)$$

If $\alpha < 3$, we divide by 0 in an intermediate step at points $x$ where $u(t, x) = 0$. To overcome this difficulty, for $0 < \delta < 1$ and $t \in J(u_0)$ we define $u_\delta(t) = (|u(t)|^2 + \delta)^\frac{1}{2}$. As above we compute

$$\mu \int_{\mathbb{R}^d} |u(t)|^{\alpha-1} \gamma \text{ div}(\ell |u(t)|^2) dx = \lim_{\delta \to 0} \mu \int_{\mathbb{R}^d} |u_\delta(t)|^{\alpha-1} \gamma \text{ div}(\ell |u(t)|^2) dx$$

$$= -\lim_{\delta \to 0} \mu \int_{\mathbb{R}^d} \left( (\alpha - 1)|u_\delta(t)|^{\alpha-3} |u(t)|^2 \gamma \text{ Re}(\pi(t) \nabla u(t) \cdot \ell) \right.$$

$$+ |u_\delta(t)|^{\alpha-1} |u(t)|^2 \ell \cdot \nabla \gamma dx$$

$$= -\mu \int_{\mathbb{R}^d} |u(t)|^{\alpha-1} \left( (\alpha - 1) \gamma \text{ Re}(\pi(t) \nabla u(t) \cdot \ell) + \ell \cdot \nabla \gamma \right) dx,$$

where the limits exist by dominated convergence thanks to $|u_\delta(t)|^{\alpha-1} \leq c(|u(t)|^{\alpha-1} + 1)$ for a constant $c > 0$, $|u_\delta(t)|^{\alpha-3} \leq |u(t)|^{\alpha-3}$ and the decay of $\gamma$ and $\nabla \gamma$. We thus obtain (14.12) as for $\alpha \geq 3$. Plugging (14.12) into (14.11), we conclude

$$\text{Re} \int_{\mathbb{R}^d} \mu |u(t)|^{\alpha-1} u(t) \left[ 2 \gamma \ell \cdot \nabla \pi(t) + (d \gamma + \ell \cdot \nabla \gamma) \pi(t) \right] dx$$

$$= \frac{\alpha - 1}{\alpha + 1} \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} (\ell \cdot \nabla \gamma + d \gamma) dx. \quad (14.13)$$

Equations (14.9), (14.10) and (14.13) now imply

$$\psi_{u, \epsilon}'(t) = \int_{\mathbb{R}^d} \gamma(2 |\nabla u(t)|^2 + \mu d |u(t)|^{\alpha+1}) dx$$

$$+ \text{Re} \int_{\mathbb{R}^d} \left( (d + 1) \pi(t) + 2 \ell \cdot \nabla \pi(t) \right) \nabla u(t) \cdot \nabla \gamma dx$$

$$+ \text{Re} \int_{\mathbb{R}^d} \pi(t) \nabla u(t) \cdot (D^2 \gamma) \ell dx$$

$$+ \frac{\alpha - 1}{\alpha + 1} \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} \ell \cdot \nabla \gamma dx$$

$$=: S_{1, \epsilon}(t) + S_{2, \epsilon}(t) + S_{3, \epsilon}(t) + S_{4, \epsilon}(t), \quad t \in J(u_0). \quad (14.14)$$
4) We next take the limit $\varepsilon \to 0$ in (14.14). Dominated convergence yields

$$S_{1,\varepsilon}(t) \to 2 \int_{\mathbb{R}^d} |\nabla u(t)|^2 \, dx + \frac{d(\alpha - 1)}{\alpha + 1} \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha + 1} \, dx$$

$$= 4E(u(t)) + \frac{d(\alpha - 1) - 4}{\alpha + 1} \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha + 1} \, dx$$

$$= 4E(u_0) + (d - \frac{2d + 4}{\alpha + 1}) \mu \int_{\mathbb{R}^d} |u(t)|^{\alpha + 1} \, dx =: \chi_u(t) \quad (14.15)$$

as $\varepsilon \to 0$, where we use that $E(u(t)) = E(u_0)$ by Theorem 13.2. Since $4\psi_u,\varepsilon$ should tend to $\phi''_u$, we expect that (14.3) follows from (14.14) and (14.15) if we can show that $S_{2,\varepsilon}$, $S_{3,\varepsilon}$ and $S_{4,\varepsilon}$ tend to 0 as $\varepsilon \to 0$. To this aim, observe that

$$\nabla\gamma_\varepsilon(x) = -2\varepsilon e^{-\varepsilon|x|^2} x, \quad |D^2\gamma_\varepsilon(x)| \leq c(\varepsilon + e^2|x|^2)e^{-\varepsilon|x|^2} \leq c(1 + e^{-1})\varepsilon \leq 2\varepsilon$$

for $\varepsilon > 0$, $x \in \mathbb{R}^d$ and a constant $c$. Hölder’s inequality now implies that

$$|S_{2,\varepsilon}(t)| \leq 2(d + 1)\varepsilon \|\ell\|_2 |\|u(t)\|_2| |\nabla u(t)|_2^2 + 4 \int_{\mathbb{R}^d} \varepsilon |x|^2 e^{-\varepsilon|x|^2} |\nabla u(t)|^2 \, dx$$

$$|S_{3,\varepsilon}(t)| \leq 2\varepsilon \|\ell\|_2 |\|u(t)\|_2| |\nabla u(t)|_2$$

$$|S_{4,\varepsilon}(t)| \leq 2 \int_{\mathbb{R}^d} \varepsilon |x|^2 e^{-\varepsilon|x|^2} |u(t)|^{\alpha + 1} \, dx. \quad (14.16)$$

All terms tend to 0 as $\varepsilon \to 0$, by dominated convergence with majorants $4|\nabla u(t)|^2$ and $2|u(t)|^{\alpha + 1}$. If we integrate the terms $S_{j,\varepsilon}$ over bounded open intervals $J$ with $J \subseteq J(u_0)$, we obtain the convergence in (14.15) and (14.16) in $L^1(J)$ (and not only pointwise) since $u \in C(J, L^1(J))$ and $\phi''_u \in C(J, L^{1+\alpha}(\mathbb{R}^d))$. Similarly, one sees that $4\psi_{u,\varepsilon}$ tends to $\phi''_u$ in $L^1(J)$ as $\varepsilon \to 0$ (recall (14.6)). As a result, $\phi''_u \in W^1_1(J)$ and $\phi''_u = 4\chi_u$. Since $\chi_u$ is continuous, it follows that $\phi''_u \in C^2(J(u_0))$ and $\phi''_u$ is given by (14.3), provided that $u_0 \in H^2(\mathbb{R}^d)$.

5) It remains to extend the step from 4) from $u_0 \in H^2(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ to $u_0 \in H^1_0(\mathbb{R}^d)$. Let $u_0 \in H^1_0(\mathbb{R}^d)$. We first approximate $u_0$ in $H^1_0(\mathbb{R}^d)$ by $\varphi_n \in C_c^\infty(\mathbb{R}^d)$.

To this aim, as in step 1) of the proof of Theorem D.13, we choose functions $\phi_n \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \phi_n \leq 1$ and $\|\partial_j \phi_n\|_\infty \leq \frac{\delta}{n}$ for all $n \in \mathbb{N}$ and $\phi_n \to 1$ pointwise on $\mathbb{R}^d$ as $n \to \infty$. Hence, $\phi_n u_0 \in H^1_0(\mathbb{R}^d)$ has compact support and $\phi_n u_0 \to u_0$ in $H^1_0(\mathbb{R}^d)$ by dominated convergence. Let $\varepsilon > 0$. We can thus find a function $v \in H^1_0(\mathbb{R}^d)$ with compact support such that $|\|u_0 - v\|_{1,2,\ell} \leq \frac{\varepsilon}{2}$. We next set $v_n = \psi_n^\varepsilon * v$ for the mollifiers $\psi_n^\varepsilon$ from formulas (D.2) – (D.7) with $n \in \mathbb{N}$. From (D.7) and Lemma D.6 we deduce that $v_n$ tends to $v$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$. Since $\supp v_n \subseteq \supp v + B(0,1)$, this fact also yields that $|\ell| v_n \to |\ell| v$ in $L^2(\mathbb{R}^d)$. As a result, we find a function $w \in C_c^\infty(\mathbb{R}^d)$ such that $\|u_0 - w\|_{1,2,\ell} \leq \varepsilon$.

So there exist initial values $\varphi_n \in C_c^\infty(\mathbb{R}^d)$ converging to $u_0$ in $H^1_0(\mathbb{R}^d)$ as $n \to \infty$. Let $J \subseteq J(u_0)$ be a compact interval with $0 \in J$. Due to Theorem 12.5 (d), there is an index $n_0 \in \mathbb{N}$ such that $J \subseteq J(\varphi_n)$ for all $n \geq n_0$ and the solutions $u_n = u(\cdot; \varphi_n)$ tend to $u$ in $C(J, H^1(\mathbb{R}^d))$ as $n \to \infty$. Equation (14.2)
and Hölder’s inequality yield that
\[
\|\ell|u_n(t)\|_2^2 = \|\ell|\varphi_n\|_2^2 + 4\text{ Im} \int_0^t \int_{\mathbb{R}^d} \nabla u_n(s) \cdot \nabla u_n(s) \, dx \, ds
\]
\[
\leq \sup_{n \geq n_0} \|\ell|\varphi_n\|_2^2 + 2 \sup_{s \in J, n \geq n_0} \|u_n(s)\|_{1,2}^2 + 2 \int_0^t \|\ell|u_n(s)\|_2^2 \, ds
\]
for all \( t \in J \) and \( n \geq n_0 \). Using Gronwall’s lemma, we then deduce that
\( \|\ell|u_n(t)\|_2 \) is uniformly bounded for \( n \geq n_0 \) and \( t \in J \). There thus exists a subsequence \((\ell u_{n_j})_j\) that converges to a function \( v \) weak* in \( L^\infty(J, L^2(\mathbb{R}^d))^d \).

Recall that \( u_n \to u \) in \( C(J, H^1(\mathbb{R}^d)) \). Considering functions \( w \in L^1(J, L^2(\mathbb{R}^d))^d \) such that \( \text{supp } w(t) \subseteq B(0, r) \) for some \( r > 0 \) and all \( t \in J \), we see that \( v = \ell u \).

The right-hand side of (14.17) (with \( u_{n_j} \) instead of \( u_n \)) thus tends to the term
\[
\|\ell|u_0\|_2^2 + 4\text{ Im} \int_0^t \int_{\mathbb{R}^d} \nabla u(s) \cdot \nabla u(s) \, dx \, ds,
\]
as \( j \to \infty \), which is equal to \( \|\ell|u(t)\|_2^2 \) due to (14.5). For each fixed \( t \in J \), the functions \( \ell|u_{n_j}(t) \) tend to \( \ell|u(t) \) a.e. on \( \mathbb{R}^d \) as \( j \to \infty \) (possibly after passing to another subsequence). Hence, \((\ell|u_{n_j}(t))_j \) tends to \( \ell|u(t) \) in \( L^2(\mathbb{R}^d) \) as \( j \to \infty \) for each \( t \in J \), where we again use Lemma 1.32 of [Kal02]. On the other hand, (14.3) for \( \varphi_{n_j} \) yields
\[
\|\ell|u_{n_j}(t)\|_2^2 = \|\ell|\varphi_{n_j}\|_2^2 + 4t \text{ Im} \int_{\mathbb{R}^d} \nabla \varphi_{n_j} \cdot \nabla \varphi_{n_j} \, dx
\]
\[
+ \int_0^t \int_0^s \left(16E(\varphi_{n_j}) + (4d - 8\frac{d+2}{\alpha+1}) \mu \|u_{n_j}(r)\|_{\alpha+1}^2\right) \, dr \, ds.
\]
We can now pass to the limit \( j \to \infty \) and obtain
\[
\phi_u(t) = \phi_u(0) + 4t \text{ Im} \int_{\mathbb{R}^d} \nabla \varphi_0 \cdot \nabla u_0 \, dx
\]
\[
+ \int_0^t \int_0^s \left(16E(u_0) + (4d - 8\frac{d+2}{\alpha+1}) \mu \|u(r)\|_{\alpha+1}^2\right) \, dr \, ds
\]
for \( t \in J \), which implies the assertion. \( \square \)

In the above proof we used the following density result to approximate a given function in two norms simultaneously. These norms involve differentiability in time and in space, respectively.

**Lemma 14.2.** Let \( J = [a, b] \) be a compact interval and let \( \mathcal{G}(J) = C^1(J, L^2(\mathbb{R}^d)) \cap C(J, H^1(\mathbb{R}^d)) \) be endowed with the norm given by \( \|u\|_{\mathcal{G}} = \max\{\|u\|_{C^1(J, L^2)}, \|u\|_{C(J, H^1)}\} \). Then the space \( C^1(J, H^1(\mathbb{R}^d)) \) is dense in \( \mathcal{G}(J) \).

**Proof.** Let \( u \in \mathcal{G}(J) \). There are functions \( \varphi_1 \in C^1([a-1, a]) \) and \( \varphi_2 \in C^1([b, b+1]) \) such that the extension \( \tilde{u} \) of \( u \) given by
\[
\tilde{u}(t) = \begin{cases} 
\varphi_1(t)u(a), & t \in [a-1, a), \\
u(t), & t \in [a, b), \\
\varphi_2(t)u(b), & t \in (b, b+1], 
\end{cases}
\]
It is straightforward to show that \( \psi_\frac{1}{n} \rightarrow u \) locally uniformly for \( t \) for all \( C_u \). For \( u \) belongs to \( \mathcal{G}([a-1, b+1]) \) and has compact support in \( (a-1, b+1) \). Extend \( \tilde{u} \) by zero to \( \mathbb{R} \). Let \( \psi_\frac{1}{n} \), \( n \in \mathbb{N} \), be a one-dimensional mollifier as in formula (D.2). For \( f \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \) we define

\[
\psi_\frac{1}{n} * f(t) = \int_{\mathbb{R}} \psi_\frac{1}{n}(t-s) f(s) \, ds = \int_{\mathbb{R}} \psi_\frac{1}{n}(s) f(t-s) \, ds, \quad t \in \mathbb{R}.
\]

It is straightforward to show that \( \psi_\frac{1}{n} * f(t) \rightarrow f(t) \) in \( L^2(\mathbb{R}^d) \) as \( n \rightarrow \infty \), locally uniformly for \( t \in \mathbb{R} \). We set \( u_n = \psi_\frac{1}{n} * \tilde{u} \). One easily sees that \( u_n \in C^1(\mathbb{R}, H^1(\mathbb{R}^d)) \), \( \nabla u_n = \psi_\frac{1}{n} * \nabla \tilde{u} \) and \( u'_n = \psi_\frac{1}{n} * \tilde{u}' \). These facts imply that \( u_n \rightarrow u \) in \( C([a, b], H^1(\mathbb{R}^d)) \) and \( C^1([a, b], L^2(\mathbb{R}^d)) \).

In our main result we will need a somewhat differently formulated version of the above proposition which is stated in the next corollary. We set \( \theta_t(x) = e^{-\frac{|x|^2}{4t}} \) for \( x \in \mathbb{R}^d \) and \( t \neq 0 \). We recall that \( E(\varphi) = \frac{1}{2} ||\nabla \varphi||^2 + \frac{\mu}{\alpha+1} ||\varphi||^\alpha_{\alpha+1} \) is the energy of \( \varphi \in H^1(\mathbb{R}^d) \).

**Corollary 14.3.** Let \( u_0 \in H^1_t(\mathbb{R}^d) \), \( \mu \in \{-1, 1\} \), \( \alpha \in (1, \alpha_c) \) and let \( u \in C^1(J(u_0), H^{-1}(\mathbb{R}^d)) \cap C(J(u_0), H^1(\mathbb{R}^d)) \) be the solution of (13.1). Set \( v(t) = \theta_t u(t) = e^{-\frac{|x|^2}{4t}} u(t) \) for \( t \in J(u_0) \setminus \{0\} \). We then have

\[
\begin{align*}
\mathcal{H}_u(t) &:= \bigg( ||(\ell + 2it
abla)u(t)||^2 \bigg) + \frac{8t^2}{\alpha + 1} \mu ||u(t)||^\alpha_{\alpha+1} \\
&= ||(\ell |u_0||^2 + \int_0^t (s(8\frac{d+2}{\alpha+1} - 4d) \mu ||u(s)||^\alpha_{\alpha+1} \, ds), \quad (14.18) \\
8t^2 E(v(t)) &:= ||(\ell |u_0||^2 + \int_0^t (s(8\frac{d+2}{\alpha+1} - 4d) \mu ||u(s)||^\alpha_{\alpha+1} \, ds \quad (14.19)
\end{align*}
\]

for all \( t \in J(u_0) \), where \( t \neq 0 \) in (14.19).

**Proof.** Since \( E(u(t)) = E(u_0) \) by Theorem 13.2, we can compute

\[
\begin{align*}
\mathcal{H}_u(t) &= ||(\ell |u(t)||^2 + 4t^2 ||\nabla u(t)||^2 + 4t \Re i \int_{\mathbb{R}^d} \nabla(t) \cdot \nabla u(t) \, dx \\
&+ \frac{8t^2}{\alpha + 1} \mu ||u(t)||^\alpha_{\alpha+1} \\
&= ||(\ell |u(t)||^2 + 8t^2 E(u_0) - 4t \Im \int_{\mathbb{R}^d} \nabla(t) \cdot \nabla u(t) \, dx
\end{align*}
\]

for \( t \in J(u_0) \). In view of Proposition 14.1, each term of the right-hand side is continuously differentiable in time so that \( h_u \in C^1(J(u_0)) \). Moreover, (14.2) and (14.3) yield

\[
\begin{align*}
\mathcal{H}_u'(t) &= 4 \Im \int_{\mathbb{R}^d} \nabla(t) \cdot \nabla u(t) \, dx + 16t E(u_0) - 4 \Im \int_{\mathbb{R}^d} \nabla(t) \cdot \nabla u(s) \, dx \\
&- 16t E(u_0) - t(4d - 8\frac{d+2}{\alpha+1}) \mu ||u(t)||^\alpha_{\alpha+1}, \\
&= t(8\frac{d+2}{\alpha+1} - 4d) \mu ||u(t)||^\alpha_{\alpha+1}.
\end{align*}
\]

Equation (14.18) follows by integration. To derive (14.19), we note that

\[
(\ell + 2it \nabla)u(t) = 2it \theta_t \nabla v(t), \quad t \in J(u_0) \setminus \{0\}.
\]

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This identity and (14.18) lead to
\[ 8t^2E(v(t)) = \|2t|\nabla v(t)\|_2^2 + \frac{8t^2}{\alpha+1} \int_{\mathbb{R}^d} |v(t)|^{\alpha+1} \, dx \]
\[ = \|\ell + 2it\nabla u(t)\|_2^2 + \frac{8t^2}{\alpha+1} \int_{\mathbb{R}^d} |u(t)|^{\alpha+1} \, dx \]
\[ = \|\ell|u_0\|_2^2 + \int_{\mathbb{R}^d} \int_0^t s(8 \frac{d+2}{\alpha+1} - 4d) \mu \|u(s)\|_{\alpha+1} \, ds \]
for \( t \in J(u_0) \setminus \{0\} \).

We can finally establish the announced convergence result in the defocusing case \( \mu = 1 \). Recall from Theorem 13.3 that in this case we have \( J(u_0) = \mathbb{R} \) for each \( u_0 \in H^1(\mathbb{R}^d) \). For simplicity we restrict ourselves to the case \( d \geq 3 \), cf. Theorem 7.3.1 in [Caz03].

**Theorem 14.4.** Let \( u_0 \in H^1_0(\mathbb{R}^d) \) and let \( u \) be the solution of the defocusing nonlinear Schrödinger equation (14.1) with \( 1 < \alpha < \alpha_c = \frac{d+2}{d-2} \) and \( d \geq 3 \). Then there is a constant \( c \) only depending on \( \alpha, d, r \) and \( \|u_0\|_{1,2} \) such that

- (a) if \( 1 + \frac{4}{d} < \alpha < \alpha_c \), then \( \|u(t)\|_r \leq c|t|^{-d(\frac{1}{2} - \frac{1}{r})} \);
- (b) if \( 1 < \alpha < 1 + \frac{4}{d} \) and \( 2 < r \leq \alpha + 1 \), then \( \|u(t)\|_r \leq c|t|^{-d(\frac{1}{2} - \frac{1}{r})} \);
- (c) if \( 1 < \alpha < 1 + \frac{4}{d} \) and \( \alpha + 1 < r \leq \alpha_c \), then \( \|u(t)\|_r \leq c|t|^{-d(\frac{1}{2} - \frac{1}{r})(1 - \theta(r))} \)

for \( |t| \geq 1 \), where \( \theta(r) = \frac{(r - \alpha - 1)(4 - d - d\alpha)}{(r - 2)(2 + d - d(\alpha - 2))} \in (0, 1) \).

**Proof.** We set \( v(t) = \theta_t u(t) \) as in Corollary 14.3. Observe that \( \|v(t)\|_p = \|u(t)\|_p \) for all \( p \in [1, \infty] \).

1) Let \( \alpha \in \left(1 + \frac{4}{d}, \alpha_c\right) \). Since \( \alpha \geq 1 + \frac{4}{d} \), the integral in (14.19) is non positive so that
\[ 2|t|\|\nabla v(t)\|_2 \leq (8t^2E(v(t)))^{\frac{1}{2}} \leq \|\ell|u_0\|_2. \] (14.20)

As in the proof of Theorem 13.3 (b), we obtain the Gagliardo-Nirenberg inequality
\[ \|\varphi\|_r \leq c_d \|\varphi\|_2^{1-d(\frac{1}{2} - \frac{1}{r})}\|\nabla \varphi\|_2^{d(\frac{1}{2} - \frac{1}{r})} \] (14.21)
for \( \varphi \in H^1(\mathbb{R}^d) \) and a constant \( c_d \) only depending on \( d \). Estimates (14.21) and (14.20) yield
\[ \|u(t)\|_r = \|v(t)\|_r \leq c_d \|v(t)\|_2^{1-d(\frac{1}{2} - \frac{1}{r})} \left(\frac{1}{2|t|}\|\ell|u_0\|_2\right)^{d(\frac{1}{2} - \frac{1}{r})} \]
\[ = c_d(2|t|)^{-\frac{d}{2} - \frac{d}{r}} \|u(t)\|_2^{-\frac{d}{2} - \frac{d}{r}} \|\ell|u_0\|_2^{-\frac{d}{2} - \frac{d}{r}} \]
\[ \leq 2^{\frac{d}{2} - \frac{d}{2} - \frac{d}{2} - \frac{d}{r}} c_d |t|^{-\frac{d}{2} - \frac{d}{r}} \|u_0\|_2 \] (14.22)
for \( t \neq 0 \), because of \( \|u(t)\|_2 = \|u_0\|_2 \) by Theorem 13.2. So (a) holds.

2) Let \( 1 < \alpha < 1 + \frac{4}{d} \). We consider the case \( t \geq 1 \). Negative times are treated in the same way. We first establish decay of \( \|v(t)\|_{\alpha+1} \) and \( \|\nabla v(t)\|_2 \). Formula (14.19) implies that
\[ 8t^2E(v(t)) = 8E(v(1)) + (8 \frac{d+2}{\alpha+1} - 4d) \int_1^t s \|u(s)\|_{\alpha+1} ds \]

for \( t \in J(u_0) \setminus \{0\} \).
for $t \geq 1$. We set $g_u(t) = t^2 \|u(t)\|_{\alpha + 1}^{\alpha + 1} = t^2 \|v(t)\|_{\alpha + 1}^{\alpha + 1}$ for $t \in \mathbb{R}$. It follows
\[ g_u(t) \leq t^2 (1 + \alpha) E(v(t)) = (1 + \alpha) E(v(1)) + \frac{1}{2} (d + 4 - da) \int_1^t \frac{1}{s} g_u(s) \, ds. \]
Since $E(u(s)) = E(u_0)$ by Theorem 13.2, equation (14.19) leads to
\[ 8 E(v(1)) \leq \|u_0\|_{2, \ell}^2 + c \sup_{0 \leq s \leq 1} \|u(s)\|_{\alpha + 1}^{\alpha + 1} \leq \|u_0\|_{2, \ell}^2 + c \sup_{0 \leq s \leq 1} E(u(s)) \]
\[ = \|u_0\|_{2, \ell}^2 + c E(u_0) \leq \|u_0\|_{2, \ell}^2 + c (\|u_0\|_{1, 2}^2 + \|u_0\|_{1, 2}^{\alpha + 1}). \]
We thus arrive at
\[ g_u(t) \leq c + \frac{1}{2} (d + 4 - da) \int_1^t \frac{1}{s} g_u(s) \, ds. \]
Here and below $c$ denotes differing constants only depending on $\alpha, d, r$ and $\|u_0\|_{1, 2}$. Set $\beta = \frac{1}{2} (d + 4 - da) > 0$. Gronwall’s inequality then yields
\[ g_u(t) \leq c \exp \int_1^t \frac{\beta}{s} \, ds = ct^\beta, \quad (14.23) \]
\[ \|v(t)\|_{\alpha + 1} = (t^{-2} g_u(t))^{\frac{1}{\alpha + 1}} \leq ct^{\frac{\beta - 2}{\alpha + 1}} = ct^{\frac{\beta - 2}{2}}. \quad (14.24) \]
for $t \geq 1$. Combining (14.19) and (14.23) we further infer
\[ 4 t^d \int_{\mathbb{R}^d} |\nabla v(t)|^2 \, dx \leq c + c \int_0^t \frac{1}{s} g_u(s) \, ds \]
\[ \leq c + c \int_0^t s^{\beta - 1} \, ds = c (1 + t^{2(\frac{2}{d} - \frac{da}{2})}), \]
\[ \|\nabla v(t)\|_2 \leq c (t^{-1} + t^{\frac{d - da}{2}}) \leq ct^{\frac{d - da}{2}}, \quad t \geq 1. \quad (14.25) \]
3) We can now show the assertions (b) and (c). We start with $r \in (2, \alpha + 1]$. Using the interpolation inequality, estimate (14.24) and $\|u(t)\|_2 = \|u_0\|_2$, we conclude
\[ \|u(t)\|_r = \|v(t)\|_r \leq \|v(t)\|_{\alpha + 1}^{\theta} \|v(t)\|^{1 - \theta}_2 \leq c t^{\theta (\frac{d - 2}{d_\alpha} - \frac{2}{d})} \|u(t)\|^{1 - \theta}_2 = ct^{\frac{d - 2}{2}} \]
for $t \geq 1$, where $\theta = (\frac{1}{2} - \frac{1}{\ell}) (\frac{1}{r} - \frac{1}{1 + \alpha})^{-1} \in (0, 1]$. Hence, (b) is valid.

To prove (c), we take $r \in (\alpha + 1, \alpha_c]$. We now employ the interpolation inequality, Sobolev’s inequality (13.9) and the estimates (14.24) and (14.25). These results lead to
\[ \|u(t)\|_r = \|v(t)\|_r \leq \|v(t)\|_{\alpha + 1}^{\theta} \|v(t)\|^{1 - \theta}_{\alpha + 1} \leq c \|\nabla v(t)\|_2^{\theta (1 - \theta)} t^{(1 - \theta)(\frac{d - 2}{d_\alpha} - \frac{2}{d})} \]
\[ \leq c t^{\theta (\frac{2}{d} - \frac{d - 2}{d_\alpha})} t^{(1 - \theta)((\frac{d - 2}{d_\alpha} - \frac{2}{d}) - (1 - \theta)(r - 1))} = ct^{\theta (\frac{2}{d} - \frac{d - 2}{d_\alpha})} (1 - \theta)(r - 1)} \]
for $\theta = (\frac{1}{r} - \frac{1}{\ell}) (\frac{1}{1 + \alpha} - \frac{d - 2}{2d})^{-1} \in (0, 1]$ and $t \geq 1$. ∎

We note that there are (weaker) decay results for $u_0 \in H^1(\mathbb{R}^d)$, see Theorem 7.7.1 of [Caz03]. The above convergence Theorem 14.4 concerns the norm in $L^r(\mathbb{R}^d)$ for $r > 2$. What happens for $r = 2$? Here the behavior of the defocusing nonlinear Schrödinger equation (14.1) is even closer to the free linear Schrödinger equation. It is described by the following scattering results:
Let $d \geq 3$, $\alpha \in (1 + \frac{4}{d}, \alpha_c)$ and $u_0 \in H^1(\mathbb{R}^d)$. Then there are unique $u_{\pm} \in H^1(\mathbb{R}^d)$ such that
\[\|u(t) - T(t)u_\pm\|_{1,2} = \|T(-t)u(t) - u_\pm\|_{1,2} \longrightarrow 0 \quad \text{as} \quad t \rightarrow \pm \infty.\]

Moreover, $\|u_\pm\|_2 = \|u_0\|_2$ and $\|\nabla u_\pm\|_2^2 = 2E(u_0)$. Finally, the maps
\[U_\pm : H^1(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d), \quad u_0 \mapsto u_\pm,\]
are continuous, bijective and have continuous inverses $\Omega_\pm : H^1(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)$ which are called “wave operators”. We refer to Section 7.8 of [Caz03] for these and related results and for references to the original papers. See also Exercise 14.3. There is an analogous scattering theory in the space $H^1_\ell(\mathbb{R}^d)$, see Section 7.4 in [Caz03].

In the focusing case $\mu = -1$, the asymptotic behavior of solutions is completely different. We have noted before that blow-up occurs if $\alpha \geq 1 + \frac{4}{d}$. (See Example 10.3 and Sections 6.5 and 6.6 of [Caz03].) Moreover, for $1 < \alpha < \alpha_c$ and $\omega > 0$ there are functions $0 \neq \varphi_\omega \in H^2(\mathbb{R}^d)$ such that
\[-\Delta \varphi_\omega + \omega \varphi_\omega = |\varphi_\omega|^{\alpha-1}\varphi_\omega,\]
which are called “wave operators”. We refer to Section 7.8 of [Caz03] for these and related results and for references to the original papers. See also Exercise 14.3. There is an analogous scattering theory in the space $H^1_\ell(\mathbb{R}^d)$, see Section 7.4 in [Caz03].

Recently, the description of the blow-up case $\alpha \geq 1 + \frac{4}{d}$ and $\mu = -1$ was much refined by W. Schlag and M. Beceanu for the model case $\alpha = d = 3$. In somewhat differing settings, they constructed a manifold of finite codimension consisting of solutions of (13.1) which converge to a variant of the manifold $S$ above, see [Bec12], [Sch09].
Exercises

Exercise 14.1. Let $1 + \frac{4}{d} \leq \alpha < \alpha_c$. Let $u_0 \in H^1(\mathbb{R}^d)$ satisfy $|x| u_0 \in L^2(\mathbb{R}^d)$ and $E(u_0) < 0$. Let $u$ be the solution of the focusing nonlinear Schrödinger equation (13.1) with $\mu = -1$. Show that the maximal existence interval $J(u_0)$ is bounded.

Exercise 14.2. Let $\alpha_0 > 0$ satisfy $d \alpha_0^2 - (d + 2) \alpha_0 - 2 = 0$, $\max\{\alpha_0, 1\} < \alpha < \alpha_c$ and $d \geq 3$. Let $u_0 \in H^1(\mathbb{R}^d)$ with $|x| u_0 \in L^2(\mathbb{R}^d)$ and let $u$ be the solution of the defocusing nonlinear Schrödinger equation (14.1) on $\mathbb{R}$. Let $p = 1 + \alpha$ and $q$ with $\frac{2}{q} + \frac{d}{p} = \frac{d}{2}$. Show that $u \in L^q(\mathbb{R}, W^1_q(\mathbb{R}^d))$.

Exercise 14.3. In the setting of Exercise 14.2 set $v(t) = T(-t)u(t)$ for $t \in \mathbb{R}$, where $T(\cdot)$ is the free Schrödinger group. Show that $v(t)$ converges in $H^1(\mathbb{R}^d)$ as $t \to +\infty$ and as $t \to -\infty$. (Hint: You may use the equation $v(t) - v(\tau) = \int_\tau^t v'(s) \, ds$ as a starting point.)
Bibliography


