The Bochner integral and vector-valued $L^p$-spaces

In this appendix we introduce the integral of Banach space valued functions, the so-called Bochner integral, define the corresponding Lebesgue and Sobolev spaces and consider the Fourier transform on a Hilbert space valued $L^2$-space. The construction of the integral is analogous to the scalar-valued case. However, at certain points one has to be more careful with separability issues.

It is assumed that the reader is familiar with the Lebesgue integral and its properties. To prove the corresponding properties of the Bochner integral, we mostly reduce to the scalar situation.

1. The Bochner integral

The presented construction of the Bochner integral follows lecture notes by R. Denk (Konstanz).

If $Y$ is a normed vector space and $M \subset Y$, then the Borel $\sigma$-algebra $\mathcal{B}(M)$ over $M$ is the $\sigma$-algebra generated by the system of relatively open subsets of $M$. We write $\mathcal{B}_d := \mathcal{B}(\mathbb{R}^d)$ for the Borel $\sigma$-algebra over $\mathbb{R}^d$. The $d$-dimensional Lebesgue measure is denoted by $dx$. In one dimension we often write $dt$. The Lebesgue measure of $A \in \mathcal{B}_d$ is denoted by $|A|$. We define $\mathcal{N}_d := \{N \in \mathcal{B}_d | |N| = 0\}$ as the set of Borel measurable sets of measure zero. For $A \in \mathcal{B}_d$, a function $f : A \rightarrow Y$ is called (Borel-)measurable if $f^{-1}(B) \in \mathcal{B}(A)$ for all $B \in \mathcal{B}(Y)$. If $f : A \rightarrow Y$ is measurable, then $\|f\|$ is measurable as well, where $\|f\|(x) := \|f(x)\|$ for $x \in A$.

Throughout, let $E$ be a complex Banach space. A function $f : \mathbb{R}^d \rightarrow X$ is called simple, if there are $N \in \mathbb{N}$, $A_n \in \mathcal{B}_d$ and $x_n \in E$ for $n = 1, \ldots, N$ such that

$$f = \sum_{n=1}^N \mathbb{1}_{A_n} x_n.$$ 

Observe that simple functions are measurable. We start with the integral over simple functions.

**Definition F.1.** Let $f = \sum_{n=1}^N \mathbb{1}_{A_n} x_n$ be a simple function with $|A_n| < \infty$ for $n = 1, \ldots, N$. Then the Bochner integral of $f$ is defined by

$$\int_{\mathbb{R}^d} f dx = \int_{\mathbb{R}^d} f(x) dx := \sum_{n=1}^N |A_n| x_n \in E.$$
We note that the above integral is independent of the representation of the simple function \( f \). It is further clear that the Bochner integral is linear on the vector space of simple functions whose support has finite measure. Moreover, as a consequence of the triangle inequality, for each simple function \( f \) we have the estimate

\[
\left\| \int_{\mathbb{R}^d} f \, dx \right\| \leq \int_{\mathbb{R}^d} \| f \| \, dx,
\]

where the integral on the right-hand side is now the usual scalar-valued Lebesgue integral.

As in the scalar case, we extend the Bochner integral to a larger class of function by taking limits of simple functions. As it turns out, besides measurability for this procedure a separability condition is necessary.

**Lemma F.2.** Let \( f : \mathbb{R}^d \to E \) be a map. Then the following assertions are equivalent.

(a) There is a sequence \( (f_k)_{k \in \mathbb{N}} \) of simple functions \( f_k : \mathbb{R}^d \to E \) such that \( f_k(x) \to f(x) \) as \( k \to \infty \) for all \( x \in \mathbb{R}^d \).

(b) \( f \) is measurable and \( f(\mathbb{R}^d) \subseteq E \) is separable.

If one of the assertions is true, then in (a) one can choose \( (f_k)_{k \in \mathbb{N}} \) such that \( \| f_k(x) \| \leq 2 \| f(x) \| \) for all \( x \in \mathbb{R}^d \).

**Proof.** Assume that (a) holds. Then \( f \) is measurable as a pointwise limit of measurable functions. Write \( f_k = \sum_{n=1}^{N_k} 1_{A_{n,k}} x_{n,k} \). Then the rational linear hull of \( \{ x_{n,k} | k \in \mathbb{N}, n = 1, ..., N_k \} \) is countable and dense in \( f(\mathbb{R}^d) \). Hence \( f(\mathbb{R}^d) \) is separable and (b) follows.

Now suppose that (b) is holds true. Let \( \{ y_k | k \in \mathbb{N} \} \) be a dense subset of \( f(\mathbb{R}^d) \). Enlarging this set by an at most countable number of vectors if necessary, we may assume that \( y_k \neq 0 \) for all \( k \). For \( k, N \in \mathbb{N} \) define

\[
\tilde{A}_k^N := \{ x \in \mathbb{R}^d | \| f(x) \| \geq 1/N, \| f(x) - y_k \| < 1/N \}.
\]

As an intersection of Borel sets the \( \tilde{A}_k^N \) are Borel sets as well. Fixing \( N \), we obtain a disjoint decomposition \( (\tilde{A}_k^N)_{k \in \mathbb{N}} \) of \( \bigcup_{k \in \mathbb{N}} \tilde{A}_k^N \) by setting

\[
A_k^N := \tilde{A}_k^N, \quad A_k := \tilde{A}_k^N \setminus \bigcup_{j=1}^{k-1} A_j^N, \quad k \in \mathbb{N}.
\]

Indeed, we have \( x \in A_k^N \) for some \( x \in \bigcup_{k \in \mathbb{N}} \tilde{A}_k^N \) if and only if \( k_0 \) is the smallest number such that \( x \in \tilde{A}_{k_0}^N \). Since \( \{ y_k | k \in \mathbb{N} \} \) is dense in \( f(\mathbb{R}^d) \), for each \( N \) we have

\[
\bigcup_{k \in \mathbb{N}} A_k^N = \bigcup_{k \in \mathbb{N}} \tilde{A}_k^N = \{ x \in \mathbb{R}^d | \| f(x) \| \geq 1/N \}.
\]

Now we can define the desired sequence \( (f_N)_{N \in \mathbb{N}} \) of simple functions. For \( N \in \mathbb{N} \) we set

\[
f_N(x) := 0 \quad \text{if} \quad x \notin \bigcup_{M,k=1}^{N} A_k^M, \quad f_N(x) := y_{k_N,x} \quad \text{if} \quad x \in \bigcup_{M,k=1}^{N} A_k^M,
\]

where the number \( k_N,x \in \{ 1, ..., N \} \) is determined as follows. Take the largest integer \( M_N,x \leq N \) such that \( x \in \bigcup_{k=1}^{N} A_k^{M_N,x} \). Employing the disjointness of
this union, $k_{N,x}$ is defined as the unique number such that $x \in A_{k_{N,x}}^M$. Note that this implies $k_{N,x} \leq N$. Hence $f_N$ takes only values in $\{0, y_1, \ldots, y_N\}$. Moreover, if $x \in A_{k_{N,x}}^M$ then
\[
\|f_N(x)\| = \|y_{k_{N,x}}\| \leq \|y_{k_{N,x}} - f(x)\| + \|f(x)\| \leq 1/M_{N,x} + \|f(x)\| \leq 2\|f(x)\|,
\]
so that $\|f_N\| \leq 2\|f\|$.

Let us check that $f_N$ is measurable. First we have $f^{-1}(\{0\}) = \mathbb{R}^d \setminus \bigcup_{N,M,k=1}^N A_k^M \in \mathcal{B}_d$. Next, let $k_x \in \{1, \ldots, N\}$. Then $f_N(x) = y_{k_x}$ if either $x \in A_{k_x}^N$ or if, for some $l \in \{1, \ldots, N - 1\}$, we have that $x$ belongs to $A_{k_x}^{N-l}$ but not to $\bigcup_{k=1}^N A_k^{N-l+m}$ for all $m \in \{1, \ldots, l\}$. In other words,
\[
f^{-1}(\{y_{k_x}\}) = A_{k_x}^N \cup \bigcup_{l=1}^{N-1} \left( A_{k_x}^{N-l} \setminus \bigcup_{m=1}^l \bigcup_{k=1}^N A_k^{N-l+m} \right) \in \mathcal{B}_d.
\]

Therefore, each $f_N$ is a simple function.

We show that $f_N \to f$ pointwise as $N \to \infty$. Let $x \in \mathbb{R}^d$. First suppose that $f(x) \neq 0$. Given $\varepsilon > 0$, we choose a natural number $N_0$ such that $\frac{\|f(x)\|}{N_0} < \min\{\|f(x)\|, \varepsilon\}$. Using (F.2) and that the sets $A_k^{N_0}$, $k \in \mathbb{N}$, are disjoint, we find a unique $k_0$ such that $x \in A_{k_0}^{N_0}$. Now consider an arbitrary number $N \geq \max\{N_0, k_0\}$. Then we have $x \in \bigcup_{N,M,k=1}^N A_k^M$, since $A_{k_0}^{N_0}$ appears in this union. Further $N_0 \leq M_{N,x} \leq N$, since $M_{N,x}$ is defined as the largest number smaller than $N$ such that $x \in \bigcup_{k=1}^N A_k^{M_{N,x}}$ and $N_0$ has this property. Since $x \in A_{k_{N,x}}^{M_{N,x}}$, it follows that
\[
\|f(x) - f_N(x)\| = \|f(x) - y_{k_{N,x}}^M\| \leq 1/M_{N,x} \leq 1/N_0 \leq \varepsilon.
\]

Finally, for $x \in \mathbb{R}^d$ with $f(x) = 0$ we have $x \notin A_k^N$ for all $k$ and $N$. Therefore $f_N(x) = 0$ and $\|f(x) - f_N(x)\| \leq \varepsilon$ is trivially satisfied. We thus conclude the pointwise convergence of $f_N$ as $N \to \infty$, and (a) follows. \hfill \Box

The lemma suggest the following notion.

**Definition F.3.** A map $f : \mathbb{R}^d \to E$ is called **strongly measurable** if there is a sequence $(f_k)_{k \in \mathbb{N}}$ of simple functions $f_k : \mathbb{R}^d \to E$ such that $f_k(x) \to f(x)$ as $k \to \infty$ for all $x \in \mathbb{R}^d$.

Another characterization of strong measurability in terms of continuous functionals is given in Lemma F.9 below.

For a strongly measurable $f$ one would like to define the Bochner integral as a limit of Bochner integrals of simple functions. Fortunately, there is a simple criterion when this is possible.

**Lemma F.4.** Let $f : \mathbb{R}^d \to E$ be strongly measurable. Then the following assertions are equivalent.

(a) There is a sequence of simple functions $(f_k)_{k \in \mathbb{N}}$ such that $f_k(x) \to f(x)$ as $k \to \infty$ for each $x \in \mathbb{R}^d$ and
\[
\lim_{k \to \infty} \int_{\mathbb{R}^d} \|f_k - f\| \, dx = 0.
\]
(b) It holds that \( \int_{\mathbb{R}^d} \|f(x)\| \, dx < \infty \).

If one of the assertions is true, then the limit \( \lim_{k \to \infty} \int_{\mathbb{R}^d} f_k \, dx \) exists in \( E \) and is independent of the sequence of simple functions \((f_k)_{k \in \mathbb{N}}\) as in (a).

**Proof.** Assume that (a) is true. First note that \( \|f_k - f\|, \|f\| : \mathbb{R}^d \to \mathbb{R}_+ \) are measurable as compositions of measurable maps. Hence \( \int_{\mathbb{R}^d} \|f_k - f\| \, dx \) and \( \int_{\mathbb{R}^d} \|f\| \, dx \) are well-defined numbers, the second one possibly equal to \( \infty \). But it indeed is finite since

\[
\int_{\mathbb{R}^d} \|f\| \, dx \leq \int_{\mathbb{R}^d} \|f - f_k\| \, dx + \int_{\mathbb{R}^d} \|f_k\| \, dx < \infty
\]

for all \( k \) such that e.g. \( \int_{\mathbb{R}^d} \|f - f_k\| \, dx \leq 1 \).

Conversely, suppose that \( \int_{\mathbb{R}^d} \|f\| \, dx < \infty \). Lemma F.2 gives a sequence of simple functions \((f_k)_{k \in \mathbb{N}}\) converging pointwise \( f \) and \( \|f_k\| \leq 2\|f\| \). Now \( g_k = \frac{1}{k} (f_k - f) \) defines a sequence of measurable functions converging pointwise to \( f \) as \( k \to \infty \), dominated by the (Lebesgue-)integrable function \( 3\|f\| \). Thus \( \lim_{k \to \infty} \int_{\mathbb{R}^d} g_k \, dx = 0 \) by the (scalar) dominated convergence theorem, which shows (a).

Finally, take a sequence \((f_k)_{k \in \mathbb{N}}\) of simple functions as in (a). Using linearity and (F.1), for \( k, l \in \mathbb{N} \) we obtain

\[
\left\| \int_{\mathbb{R}^d} f_k \, dx - \int_{\mathbb{R}^d} f_l \, dx \right\| \leq \int_{\mathbb{R}^d} \|f_k - f_l\| \, dx
\leq \int_{\mathbb{R}^d} \|f_k - f\| \, dx + \int_{\mathbb{R}^d} \|f - f_l\| \, dx,
\]

which shows that \( (\int_{\mathbb{R}^d} f_k \, dx)_{k \geq 0} \) is a Cauchy sequence in \( E \). Hence \( \int_{\mathbb{R}^d} f_k \, dx \) converges in \( E \) as \( k \to \infty \). Let \((g_k)_{k \in \mathbb{N}}\) be another sequence of simple functions as in (a). If one replaces \( f_l \) by \( g_k \) in the above estimates and takes the limit \( k \to \infty \), then one obtains that \( \lim_{k \to \infty} \int_{\mathbb{R}^d} f_k \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^d} g_k \, dx \).

Now we can define integrability and the Bochner integral for a large class of functions.

**Definition F.5.** A function \( f : \mathbb{R}^d \to E \) is called Bochner integrable if it is strongly measurable and \( \int_{\mathbb{R}^d} \|f\| \, dx < \infty \). In this case one sets

\[
\int_{\mathbb{R}^d} f \, dx := \lim_{k \to \infty} \int_{\mathbb{R}^d} f_k \, dx,
\]

where \((f_k)_{k \in \mathbb{N}}\) is any sequence of simple functions as in Lemma F.4 (a). Furthermore, for \( A \in \mathcal{B}_d \) a function \( f : A \to E \) is called Bochner integrable if its extension \( f_0 \) by zero to \( \mathbb{R}^d \) is Bochner integrable, and in this case one defines

\[
\int_A f \, dx := \int_{\mathbb{R}^d} f_0 \, dx.
\]

For \( A \in \mathcal{B}_d \) one finally sets

\[
\mathcal{L}(A, E) := \{ f : A \to E \mid f \text{ is integrable} \}.
\]
Given $A \in \mathcal{B}_d$, it follows from an approximation argument and the corresponding properties of simple functions that $\mathcal{L}(A, E)$ is a vector space, that the Bochner integral is linear and that
\[
\left\| \int_A f \, dx \right\| \leq \int_A \| f \| \, dx \quad \text{for all } f \in \mathcal{L}(A, E).
\] (F.3)

2. Properties of the Bochner integral

We collect some further basic properties of the Bochner integral that are analogous to the Lebesgue integral. Essentially, there are two strategies for the proofs. The first is to reduce to the case of simple functions by approximation, and the second is to reduce to the scalar case.

**Lemma F.6.** Let $A, A_1, A_2 \in \mathcal{B}_d$ such that $A = A_1 \cup A_2$ and suppose that $f \in \mathcal{L}(A, E)$. Then $f|_{A_1} \in \mathcal{L}(A_1, E)$, $f|_{A_2} \in \mathcal{L}(A_2, E)$ and
\[
\int_A f \, dx = \int_{A_1} f|_{A_1} \, dx + \int_{A_2} f|_{A_2} \, dx.
\]
Moreover, if $N \in \mathcal{B}_d$ is such that $|N| = 0$, then $\int_N f \, dx = 0$ for all $f \in \mathcal{L}(N, E)$.

**Proof.** Let $f_0, f_1, f_2 \in \mathcal{L}(\mathbb{R}^d, E)$ be the trivial extensions of $f$, $f|_{A_1}$ and $f|_{A_2}$ to $\mathbb{R}^d$. Take a sequence of measurable simple functions such that $f_k \to f_0$ pointwise as $k \to \infty$. Then $1_{A_1} f_k$ and $1_{A_2} f_k$ are sequences of simple functions converging pointwise to $1_{A_1} f_0 = f_1$ and $1_{A_2} f_0 = f_2$, respectively. We thus obtain
\[
\int_{\mathbb{R}^d} f_0 \, dx = \lim_{k \to \infty} \int_{\mathbb{R}^d} f_k \, dx = \lim_{k \to \infty} \left( \int_{\mathbb{R}^d} 1_{A_1} f_k \, dx + \int_{\mathbb{R}^d} 1_{A_2} f_k \, dx \right)
\]
\[
= \int_{\mathbb{R}^d} f_1 \, dx + \int_{\mathbb{R}^d} f_2 \, dx,
\]
from which the first assertion follows. Now let $|N| = 0$ and $f \in \mathcal{L}(N, E)$. Let $f_0$ be the trivial extension of $f$ to $\mathbb{R}^d$ and let $(f_k)_{k \in \mathbb{N}}$ be simple functions such that $\int_{\mathbb{R}^d} \| f - f_k \| \, dx \to 0$ as $k \to \infty$. Since $\| f - f_k \| = \| f_k \|$ almost everywhere on $\mathbb{R}^d$, we have that $\int_{\mathbb{R}^d} \| f_k \| \, dx \to 0$ as $k \to \infty$, and thus
\[
\left\| \int_{\mathbb{R}^d} f \, dx \right\| = \lim_{k \to \infty} \left\| \int_{\mathbb{R}^d} f_k \, dx \right\| \leq \lim_{k \to \infty} \int_{\mathbb{R}^d} \| f_k \| \, dx = 0,
\]
where we used (F.1). \hfill \Box

**Lemma F.7.** Let $f \in \mathcal{L}(A, E)$ and let $T \in \mathcal{B}(E, Y)$ for another Banach space $Y$. Then $Tf$, defined by $(Tf)(x) := Tf(x)$ for $x \in A$, belongs to $\mathcal{L}(A, Y)$ and we have
\[
T \int_A f \, dx = \int_A Tf \, dx.
\]

**Proof.** Take simple functions $f_k$ converging to $f_0$ pointwise on $\mathbb{R}^d$ as $k \to \infty$ such that $\lim_{k \to \infty} \int_{\mathbb{R}^d} \| f_0 - f_k \| \, dx = 0$. Then $Tf_k$ is simple for each $k$, $Tf_k \to Tf_0$ pointwise as $k \to \infty$ and
\[
\int_{\mathbb{R}^d} \| Tf_0 - Tf_k \| \, dx \leq \| T \| \int_{\mathbb{R}^d} \| f_0 - f_k \| \, dx \to 0, \quad k \to \infty.
\]
The assertions now follow from the Lemmas F.2 and F.4.

**Proposition F.8 (Dominated convergence theorem).** Let $f_k \in \mathcal{L}(A, E)$ for \( k \in \mathbb{N} \) and let \( f : A \to E \) be strongly measurable such that \( f_k \to f \) pointwise almost everywhere on \( A \) as \( k \to \infty \). Suppose there is \( g \in \mathcal{L}(A, \mathbb{C}) \) with \( \|f_k\| \leq g \) for all \( k \). Then \( f \in \mathcal{L}(A, E) \) and

$$\lim_{k \to \infty} \int_A f_k dx = \int_A f dx.$$

**Proof.** It suffices to consider the case \( A = \mathbb{R}^d \). There is \( N \in \mathcal{N}_d \) such that \( g_k := \|f_k - f\| \to 0 \) pointwise on \( \mathbb{R}^d \setminus N \) as \( k \to \infty \). Since \( \|f\| \leq g \) on \( \mathbb{R}^d \setminus N \), we have \( f \in \mathcal{L}(\mathbb{R}^d, E) \) and \( \|g_k\| \leq 2\|g\| \) on \( \mathbb{R}^d \setminus N \). Thus, using (F.3) and the scalar dominated convergence theorem, we get

$$\left\| \int_{\mathbb{R}^d} f_k dx - \int_{\mathbb{R}^d} f dx \right\| \leq \int_{\mathbb{R}^d} g_k dx \to 0$$

as \( k \to \infty \).

We finish this section with Fubini’s theorem for Bochner integrals. To this end we need another characterization of strong measurability.

**Lemma F.9 (Pettis).** For \( f : \mathbb{R}^d \to E \), the following assertions are equivalent.

(a) \( f \) is strongly measurable.

(b) For each \( x^* \in E^* \), the map \( (f, x^*) : \mathbb{R}^d \to \mathbb{C} \) is measurable, and \( f(\mathbb{R}^d) \subseteq E \) is separable.

**Proof.** If \( f \) is strongly measurable, then \( (f, x^*) \) is measurable for each \( x^* \in E^* \) since \( x^* \) is continuous. The separability of \( f(\mathbb{R}^d) \) holds by definition. Let us prove that (b) implies (a). Define \( E_0 := \text{span} \{ f(\mathbb{R}^d) \} \) and let \( \{ x_k \mid k \in \mathbb{N} \} \) be dense in \( E_0 \). Given \( N \in \mathbb{N} \), define \( s_N : E_0 \to \{ x_1, \ldots, x_N \} \) by \( s_N(y) = x_{k_N, y} \), where \( k_{N, y} \) is the smallest number \( 1 \leq k \leq N \) such that

$$\|y - x_k\| = \min_{1 \leq j \leq N} \|y - x_j\|.$$

By density we have \( \min_{1 \leq j \leq N} \|y - x_j\| \to 0 \) as \( N \to \infty \), hence \( s_N(y) \to y \) for each \( y \in E_0 \) as \( N \to \infty \). Now define the function \( f_N : A \to E \) by

$$f_N(\xi) = s_N(f(\xi)), \quad \xi \in A.$$

Then \( f_N \to f \) pointwise on \( A \) as \( N \to \infty \). Moreover, for \( 1 \leq k \leq N \),

$$f_N^{-1}(x_k) = \{ \xi \in A \mid \|f(\xi) - x_k\| = \min_{1 \leq j \leq N} \|f(\xi) - x_j\| \}$$

$$\cap \{ \xi \in A \mid \|f(\xi) - x_l\| > \min_{1 \leq j \leq N} \|f(\xi) - x_j\| \text{ for } l = 1, \ldots, k - 1 \}.$$

To conclude the strong measurability of \( f \), we have to show that the sets on the right-hand side are measurable. To this end we prove that for each \( x \in E_0 \) the functions \( \xi \mapsto \|f(\xi) - x\| \) are measurable. Then (a) follows and we are finished.

We claim that there is a sequence \( (x_k^*)_{k \in \mathbb{N}} \) of unit vectors in \( E^* \) such that

$$\|y\| = \sup_{n \in \mathbb{N}} \langle y, x_k^* \rangle, \quad x \in E_0.$$
To see this, we note that for each vector \( x_k \in E^* \) such that \( \langle x_k, x_k^* \rangle = \|x_k\| \). Given \( y \in E_0 \) and \( \varepsilon > 0 \), take \( x_{k_0} \) such that \( \|y - x_{k_0}\| \leq \varepsilon \). Then
\[
\|y\| \leq \varepsilon + |\langle x_{k_0}, x_{k_0}^* \rangle| \leq 2\varepsilon + \sup_{k \in \mathbb{N}} |\langle y, x_k^* \rangle|.
\]
Since \( \varepsilon \) is arbitrary, the claim follows.

Now we can write \( \|f(\xi) - x\| = \sup_{n \in \mathbb{N}} |\langle f(\xi) - x, x_k^* \rangle| \) for \( \xi \in A \) and \( x \in E_0 \).
By assumption in (b), \( \xi \mapsto |\langle f(\xi) - x, x_k^* \rangle| \) is measurable for each \( k \). Hence \( \xi \mapsto \|f(\xi) - x\| \) is measurable, which finishes the proof. \( \square \)

For \( d = m + n \) with \( m, n \in \mathbb{N} \) we write \( x = (y, z) \in \mathbb{R}^d \) with \( y \in \mathbb{R}^m \) and \( z \in \mathbb{R}^n \). For \( f : \mathbb{R}^d \to E \), \( y_\ast \in \mathbb{R}^m \) and \( z_\ast \in \mathbb{R}^n \) we define the functions \( f^{y_\ast} : \mathbb{R}^m \to E \) and \( f^{z_\ast} : \mathbb{R}^n \to E \) by
\[
f^{y_\ast}(z) := f(y_\ast, z), \quad z \in \mathbb{R}^n, \quad f^{z_\ast} = f(y, z_\ast), \quad y \in \mathbb{R}^m.
\]

**Proposition F.10 (Fubini’s theorem).** Let \( f \in \mathcal{L}(\mathbb{R}^d, E) \). Then there are \( M \in \mathcal{N}_m \) and \( N \in \mathcal{N}_n \) such that \( f^y \in \mathcal{L}(\mathbb{R}^m, E) \) for all \( y \in \mathbb{R}^m \setminus M \) and \( f^z \in \mathcal{L}(\mathbb{R}^n, E) \) for all \( z \in \mathbb{R}^n \setminus N \). Moreover, define the maps \( F : \mathbb{R}^m \to E \) and \( G : \mathbb{R}^n \to E \) by
\[
F(y) := \int_{\mathbb{R}^n} f^y(z)dz, \quad y \in \mathbb{R}^m \setminus M, \quad G(z) := \int_{\mathbb{R}^m} f^z(y)dy, \quad z \in \mathbb{R}^n \setminus N,
\]
and equal to zero otherwise, respectively. Then \( F \in \mathcal{L}(\mathbb{R}^m, E) \) and \( G \in \mathcal{L}(\mathbb{R}^n, E) \), and further
\[
\int_{\mathbb{R}^d} f dx = \int_{\mathbb{R}^m} F(y)dy = \int_{\mathbb{R}^n} G(z)dz. \tag{F.4}
\]

Usually, with abuse of notation, one writes equation (F.4) as
\[
\int_{\mathbb{R}^d} f dx = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(y,z)dy \right) dz = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} f(y,z)dz \right) dy.
\]

**Proof.** We show that \( f^y \in \mathcal{L}(\mathbb{R}^m, E) \) for all \( y \in \mathbb{R}^m \setminus M \), where \( M \in \mathcal{N}_m \).
Let \( x_\ast \in E^\ast \) be arbitrary. Then \( \langle f, x_\ast \rangle : \mathbb{R}^d \to \mathbb{C} \) is measurable by Lemma F.9. Thus for each \( y \in \mathbb{R}^m \) the scalar function \( \langle f^y, x_\ast \rangle \) is measurable. Since the image of \( f^y \) is separable, the strong measurability of \( f^y \) follows from Lemma F.9. By Tonelli’s theorem, we further have that
\[
\int_{\mathbb{R}^n} \|f^y(z)\|dz < \infty \quad \text{for almost all } y.
\]
Hence \( \int_{\mathbb{R}^m} \|f^y(z)\|dz < \infty \) for almost all \( y \). We conclude \( f^y \in \mathcal{L}(\mathbb{R}^m, E) \) for \( y \in \mathbb{R}^m \setminus M \) with \( M \in \mathcal{N}_m \) from Lemma F.4.

We prove that \( F \in \mathcal{L}(\mathbb{R}^m, E) \). Take simple functions \( f_k \) converging pointwise to \( f \) as \( k \to \infty \). For \( y \in \mathbb{R}^m \setminus M \), each \( \int_{\mathbb{R}^n} f_k^y(z)dz \) is simple as well, and \( \lim_{k \to \infty} \int_{\mathbb{R}^n} f_k^y(z)dz = \int_{\mathbb{R}^n} f^y(z)dz \) by dominated convergence. Hence \( F \) is strongly measurable. Since
\[
\int_{\mathbb{R}^m} \|F\|dy \leq \int_{\mathbb{R}^d} \|f\|dx < \infty
\]

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\]
by (F.3) and Tonelli’s theorem, we obtain \( F \in \mathcal{L}(\mathbb{R}^m, E) \). Finally, the equality (F.4) follows from the application of continuous functionals, Lemma F.7, Fubini’s theorem in the scalar case and the Hahn-Banach theorem. The assertions for \( G \) are shown in the same way.

\[ \square \]

## 3. Vector-valued \( L^p \) and Sobolev spaces

The definition of the vector-valued Lebesgue spaces is analogous to the scalar case.

**Definition F.11.** Let \( E \) be a Banach space and \( A \in \mathcal{B}_d \).

(a) For \( p \in [1, \infty) \) we define \( \mathcal{L}^p(A, E) \) as the set of all \( f : A \to E \) such that \( f \big|_{A \setminus N} \) is strongly measurable for some \( N \in \mathcal{N}_d \) and \( \int_{A \setminus N} \| f \|^p dx < \infty \). Moreover, for \( f \in \mathcal{L}^p(A, E) \) we set

\[
\| f \|_{\mathcal{L}^p(A, E)} := \left( \int_{A \setminus N} \| f \|^p dx \right)^{1/p}.
\]

(b) For \( p = \infty \), \( \mathcal{L}^\infty(A, E) \) is defined as the set of all \( f : A \to E \) such that \( f \big|_{A \setminus N} \) is strongly measurable for some \( N \in \mathcal{N}_d \) and

\[
\| f \|_{\mathcal{L}^\infty(A, E)} := \inf \{ c \in [0, \infty] \mid \{ \| f(x) \| > c \} = 0 \} < \infty.
\]

We observe that \( \| f \|_{\mathcal{L}^p(A, E)} \) is independent of \( N \in \mathcal{N}_d \) as above. In the same way as for \( E = \mathbb{C} \) one can prove the following facts. Part (c) below is the vector-valued version of the Fischer-Riesz theorem.

**Proposition F.12.** For \( p \in [1, \infty] \) the following holds true.

(a) For \( f, g \in \mathcal{L}^p(A, E) \) we have Minkowski’s inequality

\[
\| f + g \|_{\mathcal{L}^p(A, E)} \leq \| f \|_{\mathcal{L}^p(A, E)} + \| g \|_{\mathcal{L}^p(A, E)}.
\]

(b) For \( f \in \mathcal{L}^p(A, \mathbb{C}) \) and \( g \in \mathcal{L}^p(A, E) \), where \( p' := \frac{p}{p-1} \) for \( p \neq 1 \) and \( p':=\infty \) for \( p=1 \), we have Hölder’s inequality

\[
\| fg \|_{\mathcal{L}^{p'}(A, E)} \leq \| f \|_{\mathcal{L}^p(A, \mathbb{C})} \| g \|_{\mathcal{L}^p(A, E)}.
\]

(c) \( \mathcal{L}^p(A, E) \) endowed with \( \| \cdot \|_{\mathcal{L}^p(A, E)} \) becomes a complete semi-normed vector space under pointwise addition and scalar multiplication.

(d) If \( f_k \to f \) in \( \mathcal{L}^p(A, E) \), then there is a subsequence \( (f_{k_i})_{i \in \mathbb{N}} \) such that \( f_{k_i} \to f \) pointwise almost everywhere on \( A \).

In the usual way one obtains from \( \mathcal{L}^p(A, E) \) a normed space by considering the factor space

\[
\mathcal{L}^p(A, E) := \mathcal{L}^p(A, E) / \{ f : A \to E \mid f(x) = 0 \text{ for } dx\text{-almost every } x \in A \},
\]

i.e., by identifying functions that are equal outside a set of measure zero. Setting

\[
\| [f] \|_{\mathcal{L}^p(A, E)} := \| f \|_{\mathcal{L}^p(A, E)}
\]

for an equivalence class \([f] \in \mathcal{L}^p(A, E)\) and any \( f \in [f] \), we get that \( \| \cdot \|_{\mathcal{L}^p(A, E)} \) is a norm on \( \mathcal{L}^p(A, E) \). It is clear that \( \| [f] \|_{\mathcal{L}^p(A, E)} \) is independent of the chosen \( f \in [f] \). The completeness of \( \mathcal{L}^p(A, E) \) carries over from \( \mathcal{L}^p(A, E) \), so that \( \mathcal{L}^p(A, E) \) is a Banach space with respect to \( \| \cdot \|_{\mathcal{L}^p(A, E)} \). As usual, with a slight

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representative which can be identified with an element of $f \parallel \cdot \parallel$ in the scalar case one shows that $L^p(A, E)$ is dense in $L^p(A, E)$.

**Proof.** Let $f \in L^p(A, E)$. Since $f$ is measurable and we may assume that its image is separable, it follows from Lemma F.2 that there is a sequence of simple functions $(f_k)_{k \in \mathbb{N}}$ converging pointwise to $f$ which satisfies $\|f_k\| \leq 2\|f\|$ for all $k$. Using the dominated convergence theorem, it follows as in the proof of Lemma F.4 that $f_k \rightarrow f$ in $L^p(A, E)$.  

We finish our considerations on $L^p$-spaces by observing a density result.

**Lemma F.13.** Let $p \in [1, \infty)$. Then the set of simple functions is dense in $L^p(A, E)$. In particular, for any $p, q \in [1, \infty)$ we have that $L^p(A, E) \cap L^q(A, E)$ is dense in $L^p(A, E)$.

**Proof.** Let $f \in L^p(A, E)$. Since $f$ is measurable and we may assume that its image is separable, it follows from Lemma F.2 that there is a sequence of simple functions $(f_k)_{k \in \mathbb{N}}$ converging pointwise to $f$ which satisfies $\|f_k\| \leq 2\|f\|$ for all $k$. Using the dominated convergence theorem, it follows as in the proof of Lemma F.4 that $f_k \rightarrow f$ in $L^p(A, E)$.  

We continue with vector-valued Sobolev spaces over open intervals $(a, b) \subset \mathbb{R}$, which is sufficient for our purposes in the course.

**Definition F.14.** Let $J = (a, b)$ for $-\infty \leq a < b \leq +\infty$, let $E$ be a Banach space and let $p \in [1, \infty]$. Then the Sobolev space $W^1_p(J, E)$ is defined as the set of all $f \in L^p(J, E)$ such that there is $g \in L^p(J, E)$ satisfying

$$f(y) - f(x) = \int_x^y g(t)\, dt \quad \text{for almost every } x, y \in J.$$ 

In this case we call $g$ the weak derivative of $f$ and write $f' := g$. For $f \in W^1_p(J, E)$ we further set

$$\|f\|_{W^1_p(J, E)} := \left\{ \begin{array}{ll} \|f\|_{L^p(J, E)}^p + \|f'\|_{L^p(J, E)}^p & \text{if } p \in [1, \infty), \\ \max\{\|f\|_{L^\infty(J, E)}, \|f'\|_{L^\infty(J, E)}\} & \text{if } p = \infty. \end{array} \right.$$ 

We observe that a weak derivative is uniquely determined in $L^p(J, E)$, so that $f'$ is well-defined. It is clear that $\cdot \|_{W^1_p(J, E)}$ is a norm on $W^1_p(J, E)$. As in the scalar case one shows that $W^1_p(J, E)$ is a Banach space with respect to $\cdot \|_{W^1_p(J, E)}$. The weak derivative is a bounded linear map from $W^1_p(J, E)$ to $L^p(J, E)$.

**Proposition F.15.** Let $p \in [1, \infty]$. Then each $f \in W^1_p(J, E)$ has a unique representative which can be identified with an element of $C(\overline{J}, E)$.

**Proof.** There is $x_0 \in J$ such that $f(y) = g(y) := f(x_0) + \int_{x_0}^y f'(t)\, dt$ for almost every $y \in J$. This defines a representative $g : \overline{J} \rightarrow E$ of $f$. By the dominated convergence theorem, $g$ is continuous. The uniqueness of continuous representatives is left to the reader.  

We remark that for $f \in W^1_p(J, E)$ the above result in particular allows to give a meaning to $f(x) \in E$ for all $x \in \overline{J}$.  

4. The Fourier transform on a Hilbert space valued $L^2$-space

Let $E$ be a Hilbert space with scalar product $(\cdot | \cdot)_E$. For vector-valued integrable functions the Fourier transform is for $f \in L^1(\mathbb{R}^d, E)$ defined as in the
We extend the Fourier transform to an isometric isomorphism on $L^2(\mathbb{R}^d, E)$. Note here that $L^2(\mathbb{R}^d, E)$ is a Hilbert space with respect to the scalar product $\langle f | g \rangle_{L^2(\mathbb{R}^d, E)} := \int_{\mathbb{R}^d} (f(x) | g(x)) E dx$.

**Theorem F.16.** Let $E$ be a Hilbert space. Then $\mathcal{F}$ extends to an isometric isomorphism on $L^2(\mathbb{R}^d, E)$, which is denoted by $\mathcal{F}$ again, with inverse $\mathcal{F}^{-1}$ given by $\mathcal{F}^{-1}g(y) = (\mathcal{F}g)(-y)$ for $y \in \mathbb{R}^d$. Moreover, for $f, g \in L^2(\mathbb{R}^d, E)$ we have Plancherel’s formula

$$\langle \hat{f} | \hat{g} \rangle_{L^2(\mathbb{R}^d, E)} = \langle f | g \rangle_{L^2(\mathbb{R}^d, E)}.$$

**Proof.** Let $f = \sum_n 1_{A_n} x_n$ and $g = \sum_k 1_{B_k} y_k$ be simple functions. Using Plancherel’s theorem for the scalar-valued case, we obtain

$$\langle \hat{f} | \hat{g} \rangle_{L^2(\mathbb{R}^d, E)} = \sum_{n,k} \langle \hat{1}_{A_n} x_n | \hat{1}_{B_k} y_k \rangle_{L^2(\mathbb{R}^d, E)}$$

$$= \sum_{n,k} \langle 1_{A_n} | 1_{B_k} \rangle_{L^2(\mathbb{R}^d)} (x_n | y_k)$$

$$= \sum_{n,k} \langle 1_{A_n} | 1_{B_k} \rangle_{L^2(\mathbb{R}^d)} (x_n | y_k) = \langle f | g \rangle_{L^2(\mathbb{R}^d, E)}.$$ 

In particular, $||\hat{f}||_{L^2(\mathbb{R}^d, E)} = ||f||_{L^2(\mathbb{R}^d, E)}$ for simple functions $f$. By density (see Lemma F.13), $\mathcal{F}$ extends continuously to an isometry on $L^2(\mathbb{R}^d, E)$, and Plancherel’s formula continues to hold for the extension. Similarly, using the inversion formula in the scalar case, we obtain that $(\mathcal{F}^2f)(y) = f(-y)$ for a simple function $f$ and $y \in \mathbb{R}^d$, and this equation continues to hold on $L^2(\mathbb{R}^d, E)$. 

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Bibliography


