In this lecture we establish Strichartz’ estimates for the linear Schrödinger equation $u' = i\Delta u$ in the subcritical case. These estimates will be the crucial ingredient for our further analysis of the nonlinear Schrödinger equation. For the proof of these estimates we need Corollary 10.10 of the previous lecture, the Hardy-Littlewood-Sobolev inequality and a few other preparations. We state, discuss and show Strichartz’ estimates at the end of this lecture.

To motivate the need for Schrichartz’ estimates, we sketch the way to solve the nonlinear Schrödinger equation

$$u'(t) = i\Delta u(t) - i\mu |u(t)|^{\alpha-1}u(t), \quad t \in J, \quad u(0) = u_0. \quad (11.1)$$

Here one cannot apply the results of Lecture 8 since the nonlinearity does not map $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^d)$. We still want to use the methods of Lecture 8. In this spirit we look at the integrated version of (11.1),

$$u(t) = T(t)u_0 - i\mu \int_0^t T(t-s)u(s)|u(s)|^{\alpha-1}ds, \quad t \in J, \quad (11.2)$$

where $T(\cdot)$ is the unitary $C_0$–group generated by $i\Delta$. The right hand side of (11.2) defines an operator $\Phi$ whose fixed point will solve (11.1). To apply the contraction mapping principle, we need a function space which is mapped into itself by $\Phi$. Because of $\alpha > 1$, we loose integrability in the nonlinearity. However, the one-sided convolution with $T(\cdot)$, given by

$$T *_+ f(t) = \int_0^t T(t-s)f(s)ds, \quad t \in J,$$

could regain some integrability. There is hope for such a result since we have seen in Corollary 10.10 that $T(t)$ defined on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ has a bounded extension $T(t) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ with norm

$$\|T(t)\|_{B(L^{p'},L^p)} \leq (4\pi|t|)^{d(\frac{1}{2} - \frac{1}{p'})} = (4\pi|t|)^{\frac{d}{2} - \frac{d}{2}} \quad (11.3)$$

for all $p \in [2, \infty]$ and $t \in \mathbb{R} \setminus \{0\}$. One calls (11.3) a dispersive estimate since it says that initial values in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ are mapped to bounded functions by the free Schrödinger group $T(\cdot)$. Because the $L^2$–norm is preserved, the solution must spread out in space.

To treat the fixed point problem (11.2), we have to bound $T *_+ f$ and the orbit $T(\cdot)u_0$. The needed inequalities will follow from (11.3), but this derivation fills the entire lecture. The resulting estimates are named after R. Strichartz who proved such estimates for the wave equation in 1977 (see [Str77]). Various versions were then shown in the following years, see Section 2.3 in [Caz03]. The development culminated in the paper [KT98] from 1998 by Keel and Tao who
established the result in a certain borderline case (described below) which is
needed for the critical case $\alpha = \alpha_\text{c}$ in (11.1), cf. (10.3). By now, Strichartz
type estimates are known for a wide range of dispersive equations.

We will show a basic version of Strichartz’ estimates in the subcritical case
for the free Schrödinger group. This result fits to our problem (11.1) with
$1 < \alpha < \alpha_\text{c}$. To exploit the dispersive estimate (11.3), one needs the Hardy-
Littlewood-Sobolev inequality for the space dimension $n = 1$, see (11.4), which
is of independent interest.

To derive this inequality, we first represent the $L^p$-norm of a function $f : \mathbb{R}^n \to \mathbb{C}$ in terms of its distribution function $d_f$ defined by
\[
d_f(s) = \int_{\mathbb{R}^n} 1_{\{|f| > s\}}(x) \, dx, \quad s \geq 0.
\]
Here we use the notation $\{|f| > s\} = \{\xi \in \mathbb{R}^n \mid |f(\xi)| > s\}$.

**Lemma 11.1.** For all $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^n)$ we have
\[
\|f\|_p^p = p \int_0^\infty s^{p-1} d_f(s) \, ds.
\]

**Proof.** Using Fubini’s theorem, we compute
\[
\|f\|_p^p = \int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_{\mathbb{R}^n} \left( \int_0^{|f(x)|} s^{p-1} \, ds \right) \, dx
\]
\[
= p \int_0^\infty s^{p-1} \left( \int_{\mathbb{R}^n} 1_{\{|f(x)| \leq s\}}(s) \, dx \right) \, ds.
\]
The assertion now follows from the fact that $1_{\{0 \leq |f(x)| \leq s\}}(s) = 1_{\{|f| > s\}}(x)$. $\square$

We next show the Hardy-Littlewood-Sobolev inequality. Its proof is elementary
and laborious but quite entertaining.

**Theorem 11.2** (Hardy-Littlewood-Sobolev inequality). Let $\beta, \gamma \in (1, \infty)$
and $0 < \lambda < n$ satisfy $\frac{1}{\beta} + \frac{\lambda}{n} + \frac{1}{\gamma} = 2$. Then there is a constant $C > 0$ such that
\[
\int_{\mathbb{R}^n} |x|^\lambda \left| \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^\lambda} \, dy \right| \, dx \leq C\|f\|_\beta \|g\|_\gamma
\]
for all $f \in L^\beta(\mathbb{R}^n)$ and all $g \in L^\gamma(\mathbb{R}^n)$. As a result,
\[
\left( \int_{\mathbb{R}^n} \left[ \int_{\mathbb{R}^n} |g(y)| \frac{\beta'}{|x-y|^{\beta'}} \, dy \right]^{\gamma'} \, dx \right)^{\frac{1}{\gamma'}} \leq C\|g\|_\gamma.
\]

The inequality (11.5) resembles Young’s convolution estimate. (See Theorem
1.2.12 in [Gra08].) To see this relation, let $\varphi_\lambda(x) = |x|^{-\lambda}$ for $x \in \mathbb{R}^n \setminus \{0\}$ and
$\varphi_\lambda(0) = 0$, where $\lambda \in (0, n)$. If we had $\varphi_\lambda \in L^r(\mathbb{R}^n)$ with $\frac{1}{r} = 1 + \frac{1}{\beta} - \frac{1}{\gamma}$ and
$r, \beta, \gamma \in [1, \infty]$, then Young’s inequality would give
\[
\||\varphi_\lambda * g||_{\beta'} \leq ||\varphi_\lambda||_r \|g\|_\gamma \quad \text{for all } g \in L^\gamma(\mathbb{R}^n).
\]
To show (11.5), we would need here $\lambda = n(2 - \frac{1}{\beta} - \frac{1}{\lambda})$, i.e., $\lambda r = n$. So
$\varphi_\lambda \in L^r(\mathbb{R}^n)$ would require that $\int_{\mathbb{R}^n} |x|^{-n} \, dx < \infty$ which is not quite true. In
(11.5) we have shown (11.6) (at least for positive $g$) with $||\varphi_\lambda||_r$ replaced by $C$. 110
Proof. 1) Assertion (11.5) follows from (11.4) by a duality argument. For (11.4) it suffices to prove the inequality
\[
I(f, g) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{\lambda}} \, dx \, dy \leq C
\]
under the assumption that \( f, g \geq 0 \) and \( \|f\|_\beta = \|g\|_\gamma = 1 \). The derivation of (11.7) is based on Lemma 11.1 and the distribution functions \( d_f \) and \( d_g \). As in the proof of Lemma 11.1, for \( x \in \mathbb{R}^n \) we have
\[
f(x) = \int_0^\infty \mathbbm{1}_{\{0, f(x))}(r) \, dr = \int_0^\infty \mathbbm{1}_{\{f > r\}}(x) \, dr,
\]
and in the same way \( g(y) = \int_0^\infty \mathbbm{1}_{\{g > s\}}(y) \, ds \) for \( y \in \mathbb{R}^n \). Moreover,
\[
|x|^{-\lambda} = \lambda \int_{|x|}^\infty t^{-\lambda-1} \, dt = \lambda \int_0^\infty t^{-\lambda-1} \mathbbm{1}_{\{|x|, \infty\}}(t) \, dt = \lambda \int_0^\infty t^{-\lambda-1} \mathbbm{1}_{B(0, 1)}(x) \, dt.
\]
These facts and Fubini’s theorem yield
\[
I(f, g) = \lambda \int_0^\infty \int_0^\infty \int_0^\infty t^{-\lambda-1} J(r, s, t) \, ds \, dr,
\]
where we have set
\[
J(r, s, t) := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbbm{1}_{\{f > r\}}(x) \mathbbm{1}_{\{g > s\}}(y) \mathbbm{1}_{B(0, 1)}(x-y) \, dx \, dy.
\]
2) Let \( r, s, t > 0 \). We estimate \( J(r, s, t) \) in three ways. We first derive
\[
J(r, s, t) \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathbbm{1}_{\{g > s\}}(y) \mathbbm{1}_{B(0, 1)}(x-y) \, dx \, dy
\]
\[
= \int_{\mathbb{R}^n} \mathbbm{1}_{\{g > s\}}(y) \int_{B(y, t)} 1 \, dx \, dy = C_1 d_g(s) t^n,
\]
where \( C_1 = \text{vol}(B(0, 1)) \). Similarly, one obtains
\[
J(r, s, t) \leq C_1 d_f(r) t^n, \quad J(r, s, t) \leq d_f(r) d_g(s).
\]
3) We fix \( r, s > 0 \) and estimate the inner integral in (11.8) in two ways. We first split the integral at \( t = d_f(r)^{\frac{1}{n}} \) and use (11.9) for the first summand and the second part of (11.10) for the second summand, arriving at
\[
\int_0^\infty t^{-\lambda-1} J(r, s, t) \, dt \leq C_1 \int_0^{d_f(r)^{\frac{1}{n}}} d_g(s) t^{-\lambda-1+n} \, dt + \int_0^\infty 1_{d_f(r)^{\frac{1}{n}}} d_f(r) d_g(s) t^{-\lambda-1} \, dt
\]
\[
= \frac{C_1}{n-\lambda} d_g(s) d_f(r) \frac{n-\lambda}{n} + \frac{1}{\lambda} d_f(r) d_g(s) \frac{-1}{\lambda} d_f(r)^{1-\frac{1}{\lambda}} d_g(s).
\]
Second, splitting at \( t = d_g(s)^{\frac{1}{n}} \) and applying both parts of (11.10), we deduce
\[
\int_0^\infty t^{-\lambda-1} J(r, s, t) \, dt \leq \frac{\lambda C_1 + n - \lambda}{\lambda(n-\lambda)} d_f(r) d_g(s)^{1-\frac{1}{n}}.
\]
Inserting these inequalities into (11.8), we conclude
\[
I(f, g) \leq \frac{n(C_1 + 1)}{n - \lambda} \int_0^\infty \int_0^\infty \min\{d_f(r)^{1 - \frac{\lambda}{n}} d_g(s), d_f(r) d_g(s)^{1 - \frac{\lambda}{n}}\} \, ds \, dr.
\]

(11.11)

4) We next split the inner integral in (11.11) at \(s = \frac{\gamma}{n}\) and obtain
\[
I(f, g) \leq \frac{n(C_1 + 1)}{n - \lambda} \int_0^\infty d_f(r) \left(\int_0^{\frac{\gamma}{n}} d_g(s)^{1 - \frac{\lambda}{n}} \, ds\right) \, dr
\]
\[
+ \frac{n(C_1 + 1)}{n - \lambda} \int_0^\infty d_f(r) \left(\int_{\frac{\gamma}{n}}^\infty d_g(s)^{1 - \frac{\lambda}{n}} \, ds\right) \, dr.
\]

(11.12)

In the first summand we apply Hölder’s inequality with the exponents \((1 - \frac{\lambda}{n})^{-1} > 1\) and \(\frac{\lambda}{n} > 1\). Lemma 11.1 and \(\|g\|_\gamma = 1\) then imply
\[
\int_0^{\frac{\gamma}{n}} d_g(s)^{1 - \frac{\lambda}{n}} \, ds = \int_0^{\frac{\gamma}{n}} (d_g(s))^{1 - \frac{\lambda}{n}} s^{(\gamma - 1)(1 - \frac{\lambda}{n})} s^{-(\gamma - 1)(1 - \frac{\lambda}{n})} \, ds
\]
\[
\leq \left(\int_0^{\frac{\gamma}{n}} d_g(s) s^{\gamma - 1} \, ds\right)^{1 - \frac{\lambda}{n}} \left(\int_0^{\frac{\gamma}{n}} s^{-(\gamma - 1)(1 - \frac{\lambda}{n})} \, ds\right)^{\frac{\lambda}{n}}
\]
\[
\leq \left(\frac{1}{\gamma} \|g\|_\gamma\right)^{1 - \frac{\lambda}{n}} \left(\int_0^{\frac{\gamma}{n}} s^{-(\gamma - 1)(1 - \frac{\lambda}{n})} \, ds\right)^{\frac{\lambda}{n}} \leq C_2 \beta^{\beta - 1}
\]
for a constant \(C_2\) only depending on \(\beta, \lambda, n\), where the assumption \(\frac{1}{\beta} + \frac{\lambda}{n} + \frac{\lambda}{n} = 2\) yields \((\gamma - 1)(\frac{\lambda}{n} - 1) < 1\) as well as \(\frac{\lambda}{\gamma}(1 - (\gamma - 1)(\frac{\lambda}{n} - 1)) = \beta - 1\).

Employing again Lemma 11.1, we infer
\[
\int_0^\infty d_f(r) \left(\int_0^{\frac{\gamma}{n}} d_g(s)^{1 - \frac{\lambda}{n}} \, ds\right) \, dr \leq C_2 \int_0^\infty d_f(r) r^{\beta - 1} \, dr = \frac{C_2}{\beta^\beta} \|f\|_\beta = \frac{C_2}{\beta^\beta}.
\]

The second double integral in (11.12) can be rewritten as
\[
\int_0^\infty d_f(r) \left(\int_{\frac{\gamma}{n}}^\infty d_g(s)^{1 - \frac{\lambda}{n}} \, ds\right) \, dr
\]
\[
= \int_0^\infty \int_0^\infty 1_{(0, \frac{\gamma}{n})}(s) d_f(r)^{1 - \frac{\lambda}{n}} d_g(s) \, ds \, dr
\]
\[
= \int_0^\infty 1_{(0, s^{\gamma/\beta})}(r) d_f(r)^{1 - \frac{\lambda}{n}} d_g(s) \, drds
\]
\[
= \int_0^\infty d_g(s) \left(\int_0^{s^{\gamma/\beta}} d_f(r)^{1 - \frac{\lambda}{n}} \, dr\right) \, ds,
\]
due to Fubini’s theorem. This term can be bounded in the same way as above by a constant \(C_3\). We have thus shown (11.7).

In the proof of Strichartz’ estimates we have to commute partial derivatives with the free Schrödinger group \(T(\cdot)\), which is justified by the next lemma.

**Lemma 11.3.** For each \(k \in \mathbb{N}\), the free Schrödinger group \(T(\cdot)\) leaves \(H^k(\mathbb{R}^d)\) invariant and induces a unitary \(C_0\)-group on \(H^k(\mathbb{R}^d)\). Moreover, \(\partial_j T(t)v = T(t) \partial_j v\) for all \(t \in \mathbb{R}, j \in \{1, \ldots, d\}\) and \(v \in H^1(\mathbb{R}^d)\). An analogous result holds for higher derivatives.
PROOF. Let $t \in \mathbb{R} \setminus \{0\}$ and $k = 1$. We first show $T(t)v \in H^1(\mathbb{R}^d)$ for $v \in C_c^\infty(\mathbb{R}^d)$. Due to Lemma 10.9, there is a kernel $K_t \in C_b(\mathbb{R}^d)$ such that

$$T(t)v(x) = (K_t * v)(x) = \int_{\mathbb{R}^d} K_t(y)v(x - y) \, dy$$

for $x \in \mathbb{R}^d$. Thus $T(t)v \in C^1(\mathbb{R}^d)$ and

$$\partial_j T(t)v = \int_{\mathbb{R}^d} K_t(y)(\partial_j v)(\cdot - y) \, dy = T(t)(\partial_j v)$$

on $\mathbb{R}^d$. Next, let $v \in H^1(\mathbb{R}^d)$. By Remark 5.8 (c), there are functions $v_n \in C_c^\infty(\mathbb{R}^d)$ that converge to $v$ in $H^1(\mathbb{R}^d)$ as $n \to \infty$. Hence, $\partial_j T(t)v_n = T(t)\partial_j v_n$ tend to $T(t)\partial_j v$ in $L^2(\mathbb{R}^d)$. Remark 5.8 (b) thus yields that $T(t)v \in H^1(\mathbb{R}^d)$ and $\partial_j T(t)v = T(t)\partial_j v$ for $j \in \{1, \ldots, d\}$. It follows that $T(t)$, $t \in \mathbb{R}$, can be restricted to an isometry on $H^1(\mathbb{R}^d)$. Clearly, the restriction still satisfies the group property. Since $\|\partial_j(T(t)v - v)\|_2 = \|T(t)\partial_j v - \partial_j v\|_2 \to 0$ as $t \to 0$, the group $T(\cdot)$ is strongly continuous on $H^1(\mathbb{R}^d)$. By the group law the operators $T(t) : H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ are bijective and hence unitary due to Theorem C.7. The case $k \geq 2$ is treated similarly. \qed

Strichartz’ estimate will be formulated in the Banach space-valued Lebesgue space $L^q(J, W_p^k(\mathbb{R}^d))$ with $p, q \in [1, \infty]$, $k \in \mathbb{N}_0$ and an interval $J \subseteq \mathbb{R}$, where we recall our notations $W_p^0 = L^p$ and $H^0 = L^2$. In Appendix F we have collected the definitions and some of the basic properties of these spaces. We further need the following results.

REMARK 11.4. (a) Let $X$ be reflexive and $1 \leq q < \infty$. Then $L^q(J, X)^*$ is isometrically isomorphic to $L^q(J, X^*)$, where $g \in L^q(J, X^*)$ acts via

$$\langle f, g \rangle_{L^q(J, X)} = \int_J \langle f(t), g(t) \rangle_X \, dt$$

on $f \in L^q(J, X)$. Moreover, $L^q(J, X)$ is reflexive if $1 < q < \infty$. (See Theorems 8.20.3 and 8.20.5 in [Edw65].) \diamond

(b) Let $X$ be separable and $1 \leq q < \infty$. Then the space $L^q(J, X)$ is separable. In fact, in Lemma F.13 we have seen that the set of $X$-valued simple functions is dense in $L^q(J, X)$ if $1 \leq q < \infty$. Standard properties of the Lebesgue measure allow to approximate a simple function by sums of functions of the form $1_Q x$ for $x \in X$ and an interval $Q \subseteq \mathbb{R}^d$ with rational vertices. Inserting here a dense sequence $(x_n)$ in $X$, we obtain a dense countable subset of $L^q(J, X)$. \diamond

In some of our arguments we have to approximate a function in $L^q(J, W_p^k(\mathbb{R}^d))$ by smoother ones. In the next lemma we show a corresponding density result.

**LEMMA 11.5.** Let $J \subseteq \mathbb{R}$ be an open interval, $k \in \mathbb{N}_0$ and $1 \leq p, q < \infty$. For each $f \in L^q(J, W_p^k(\mathbb{R}^d)) =: E$ there are $\varphi_n \in \bigcap_{m \in \mathbb{N}_0, r \in [1, \infty]} C_c^\infty(J, W_r^m(\mathbb{R}^d))$ converging to $f$ in $E$ as $n \to \infty$.

**PROOF.** Let $f \in E$, $m \in \mathbb{N}_0$ and $r \in [1, \infty]$. Let $\varepsilon > 0$. We proceed in four steps involving cutoff and mollification in space and time, cf. Appendix D.

1) Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be equal to 1 on $\overline{B}(0, 1)$ and have support in $B(0, 2)$. Set $\phi_n(x) = \phi(\frac{x}{n})$. Then $\phi_n = 1$ on $\overline{B}(0, n)$, supp $\phi_n \subseteq B(0, 2n)$ and all derivatives
of \( \phi_n \) of order \( k \in \mathbb{N}_0 \) are uniformly bounded by \( n^{-k} \| \phi \|_{C^k} \). As a result, \( \phi_n f(t) \) belongs to \( W^k_p(\mathbb{R}^d) \) and has support in \( B(0,2n) \) for each \( n \in \mathbb{N} \) and a.e. \( t \in J \). Moreover, the functions \( \phi_n f(t) \) tend to \( f(t) \) in \( W^k_p(\mathbb{R}^d) \) as \( n \to \infty \), for a.e. \( t \in J \). (See part 1) of the proof of Theorem D.13.) Lebesgue’s theorem (with majorant \( c_k \| f(t) \|_{k,p} \)) thus shows that \( \phi_n f \to f \) in \( E \). We can now fix \( N \) such that \( g = \phi_N f \in E \) satisfies \( \| f - g \|_E \leq \varepsilon \) and \( \text{supp}(g(t)) \subseteq B(0,2N) \) for a.e. \( t \in J \).

2) We next use the mollifiers \( G_n \) on \( W^k_p(\mathbb{R}^d) \) given by \( G_n v = \rho_n \ast v \), where \( \rho_n(x) = n^{-d} |x|^{-1} \chi(nx) \) for \( n \in \mathbb{N} \) and \( x \in \mathbb{R} \) as well as \( \chi(x) = \exp(-\frac{1}{2|x|^2 - 1}) \) for \( |x| < 1 \) and \( \chi(x) = 0 \) for \( |x| \geq 1 \), see (D.3). If \( v \in L^1(\mathbb{R}^d) \) has support in \( B(0,R) \), one easily checks that \( \text{supp} G_n v \subseteq B(0,R+1) \), \( G_n v \in C^\infty(\mathbb{R}^d) \) and \( \| G_n v \|_{m,r} \leq C(m,r,R) \| v \|_1 \) for all \( n \in \mathbb{N} \) and a constant only depending on \( m, r, R \) and \( \chi \). Hence, \( G_n g \in L^q(J, W^m_p(\mathbb{R}^d)) \). The standard properties of \( G_n \) stated in (D.6) – (D.7) and Lemma D.6 (a) imply that \( \| G_n(g(t)) \|_{k,p} \leq \| g(t) \|_{k,p} \) for all \( n \in \mathbb{N} \) and \( G_n g(t) \to g(t) \) in \( W^m_p(\mathbb{R}^d) \) as \( n \to \infty \) for a.e. \( t \in J \). As in step 1), we can find an index such that \( h = G_N g \in L^q(J, W^m_p(\mathbb{R}^d)) \cap E \) satisfies \( \| g - h \|_E \leq \varepsilon \).

3) Let \( J_n \subseteq J \subseteq J_{n+1} \subseteq J \) be open bounded intervals with \( \bigcup_{n \in \mathbb{N}} J_n = J \). There are functions \( \psi_n \in C_c(J) \) with \( 0 \leq \psi_n \leq 1 \), \( \psi_n = 1 \) on \( J_n \) and \( \text{supp} \phi_n \subseteq J_{n+1} \). Lebesgue’s theorem gives an index \( N \) such that \( \psi = \phi_N h \in L^q(J, W^m_p(\mathbb{R}^d)) \cap E \), \( \| \psi - h \|_E \leq \varepsilon \) and \( \psi \) has compact support in \( J \).

4) Finally, we apply the mollifier \( G^1_n \psi = \rho^1_n \ast \tilde{\psi} \) from step 2) with \( d = 1 \), where \( \tilde{\psi} \) is the 0-extension of \( \psi \) to \( \mathbb{R} \) and we restrict \( G^1_n \psi \) to \( J \). It is straightforward to check that \( G^1_n \psi \in C^\infty_c(J, W^m_p(\mathbb{R}^d)) \cap E \). The usual properties of mollifiers also work in the Banach space-valued case and we obtain an index \( N \) and \( \varphi = \rho^1_N \ast \psi \in C^\infty_c(J, W^m_p(\mathbb{R}^d)) \) such that \( \| \psi - \varphi \|_E \leq \varepsilon \).

Strichartz’ estimates involve the spaces \( L^q(J, W^k_p(\mathbb{R}^d)) \) for certain pairs of admissible exponents \((q,p)\). This means that

\[
\frac{2}{q} + \frac{d}{p} = \frac{d}{2} \quad \text{where} \quad 2 \leq q \leq \infty \quad \text{and} \quad \begin{cases} 1 \leq p < \infty, & d \geq 2, \\ 1 \leq p \leq \infty, & d = 1. \end{cases} \tag{11.13}
\]

We see in Remark 11.7 that this relation between \( p \) and \( q \) is determined by scaling properties of the linear Schrödinger equation. For other dispersive equations (e.g. the wave equation) one obtains a different concept of admissibility (see e.g. Section 2.3 in [Tao06]).

We can visualize the reciprocals \((\frac{1}{q}, \frac{1}{p})\) of an admissible pair as the line segment between the reciprocals of \((\infty, 2)\) and \((2, \frac{2d}{d-2})\) if \( d \geq 2 \), where \((2, \infty) = (2, \frac{2d}{d-2})\) is excluded for \( d = 2 \). In the case \( d = 1 \) one has the line between the inverses of \((\infty, 2)\) and \((4, \infty)\).

We remark that for admissible \((q,p)\) we have \( H^1(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d) \) and thus \( L^p(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d) \), since \( 1 - \frac{d}{2} \geq \frac{2}{q} - \frac{d}{2} = -\frac{d}{p} \), see Sobolev’s embedding (5.5). As a result, for admissible \((q,p)\) and \( f \in L^q(J, L^p(\mathbb{R}^d)) \) the one-sided convolution

\[
T \ast_+ f(t) = \int_0^t T(t-s)f(s) \, ds, \quad t \in J,
\]
is defined as a Bochner integral in $H^{-1}({\mathbb R}^d)$. Recall that the free Schrödinger group $T(\cdot)$ is strongly continuous on $H^{-1}({\mathbb R}^d)$ by Lemma 10.7.

We now state our main theorem. Part (a) is called the homogeneous and (b) the inhomogeneous Strichartz’ estimate.

**Theorem 11.6 (Strichartz’ estimates for $i\Delta$).** Let $(q,p)$ and $(\overline{q},\overline{p})$ be admissible, $k \in N_0$, $\varphi \in H^k({\mathbb R}^d)$, $J \subseteq {\mathbb R}$ be an interval with $0 \in J$, and $f \in L^\overline{p}(J,W^k_p({\mathbb R}^d))$. Then $T \ast_+ f(t)$ exists in $W^k_p({\mathbb R}^d)$ for a.e. $t \in J$, $T(\cdot)\varphi$ and $T \ast_+ f$ belong to $L^q(J,W^k_p({\mathbb R}^d))$ and

\[
(a) \|T(\cdot)\varphi\|_{L^q(J,W^k_p({\mathbb R}^d))} \leq c \|\varphi\|_{k,2},
\]

\[
(b) \|T \ast_+ f\|_{L^q(J,W^k_p({\mathbb R}^d))} \leq c \|f\|_{L^\overline{p}(J,W^k_p({\mathbb R}^d))}
\]

for a constant $c > 0$ (independent of $\varphi$, $f$, $k$ and $J$). If $q = \infty$ and $p = 2$, we can replace $L^\infty$ by $C_0$ in (a) and (b).

Compared to the $L^2$-setting, in the above estimates one gains space integrability from $p = 2$ to $p > 2$, but one loses time integrability from $q = \infty$ to $q < \infty$. Similarly, in (b) the exponents on the right–hand side are smaller than 2, whereas they are larger than 2 on the left–hand side. We point out that $(q,p)$ can be chosen independently of $(\overline{q},\overline{p})$ in Theorem 11.6 (b).

We will prove Theorem 11.6 only for $q, \overline{q} > 2$ and either for $(q,p) = (\overline{q},\overline{p})$ or for $(q,p) = (\infty,2)$ and any admissible $(\overline{q},\overline{p})$, since we only work with these cases later on. Exponents $(q,p) \neq (\overline{q},\overline{p})$ are needed for certain more general nonlinearities in (11.1). This case requires another tool, the Christ-Kiselev lemma, see Section 2.3 in [Tao06]. The endpoint case $(2, \frac{2d}{d-2})$ for $d \geq 3$ is much more difficult, see [KT98]. It is needed to study (11.1) for the critical case $\alpha = \alpha_c$.

In the next proof we use that $X \cap Y$ is a Banach space for the norm given by $\|x\|_X + \|y\|_Y$ for $x \in X$ and $y \in Y$, where $X$ and $Y$ are Banach spaces.

**Proof of 11.6 (for $q, \overline{q} > 2$ and either $(q,p) = (\overline{q},\overline{p})$ or $(q,p) = (\infty,2)$).**

We first consider $J = {\mathbb R}$ and $k = 0$. The other cases and the final assertion are treated afterwards.

1) Let $(q,p) = (\overline{q},\overline{p})$ be admissible, $2 < q < \infty$, $\varphi \in L^2({\mathbb R}^d)$ and $f \in L^\overline{p}({\mathbb R},L^p({\mathbb R}^d)) =: E'$. Set $E = L^q({\mathbb R},L^p({\mathbb R}^d))$. We first prove (b). Inequality (11.3) and Theorem 11.2 (with $\lambda = \frac{2}{p} - \frac{d}{2}$, $n = 1$, $\beta = \gamma = q'$) imply the crucial estimate

\[
I_1 := \left[\int_{{\mathbb R}} \left[\int_0^t \|T(t-s)f(s)\|_p \, ds\right]^q \, dt\right]^\frac{1}{q} \leq \left[\int_{{\mathbb R}} \left[\int_{{\mathbb R}} (4\pi|t-s|)^{\frac{d}{2}} \|f(s)\|_p \, ds\right]^{\frac{q}{p}} \, dt\right]^\frac{1}{q} \leq C_0 \left[\int_{{\mathbb R}} \|f(s)\|_{\overline{p}}^q \, ds\right]^{\frac{1}{q}}
\]

(11.14)

where $C_0$ only depends on $d$, $p$ and $q$. The conditions of Theorem 11.2 hold since $(q,p)$ is admissible and $2 < q < \infty$.

From this estimate assertion (b) will follow by means of Fubini’s theorem, but the details concerning integrability are a bit tricky. To this aim, take $m \in {\mathbb N}$ with $m \geq \frac{d}{2} - \frac{d}{p}$ so that $H^m({\mathbb R}^d) \hookrightarrow L^p({\mathbb R}^d)$ by Sobolev’s embedding (5.5).
Lemma 11.5 yields \( g_n \in C_c(\mathbb{R}, H^m(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)) \) that converge to \( f \) in \( E' \). The function
\[
\mathbb{R}^2 \to L^p(\mathbb{R}^d), \quad (t, s) \mapsto T(t - s)g_n(s),
\]
is continuous for each \( n \in \mathbb{N} \) since it is continuous in \( H^m(\mathbb{R}^d) \) by Lemma 11.3. There is a subsequence such that the functions \( g_{n_j}(s) \) converge in \( L^{p'}(\mathbb{R}^d) \) to \( f(s) \) as \( j \to \infty \) for a.e. \( s \in \mathbb{R} \). Moreover, \( T(t - s) \) maps \( L^{p'}(\mathbb{R}^d) \) continuously into \( L^p(\mathbb{R}^d) \) for \( t \neq s \), see (11.3). Therefore \((t, s) \mapsto T(t - s)f(s)\) is strongly measurable with values in \( L^p(\mathbb{R}^d) \), outside a set of measure 0. One further sees that \( T(\cdot) \varphi : \mathbb{R} \to L^p(\mathbb{R}^d) \) is strongly measurable if \( \varphi \in L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d) \).

It now follows from Fubini’s theorem and (11.14) that the integral \((T * f)(t)\) exists in \( L^p(\mathbb{R}^d) \) for a.e. \( t \in \mathbb{R} \) and that \( T * f : \mathbb{R} \to L^p(\mathbb{R}^d) \) is strongly measurable. Since \( \|T * f\|_E \leq I_1 \), assertion (b) holds. In the same way one sees that \( \|T * f\|_E \leq C_0 \|f\|_E \) for the usual convolution.

2) We show (a) by a duality argument in the framework of step 1). We first consider \( g \in C_c(\mathbb{R}, L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)) \). Remark 11.4 and step 1) imply
\[
I_2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} T(t - s)g(s) \, ds \right) \overline{g(t)} \, dx \, dt = \langle T * g, \overline{g} \rangle_{L^q(\mathbb{R}^d)} = \|T * g\|_{L^q(\mathbb{R}^d)} \|g\|_{L^{p'}(\mathbb{R}^d)}^2 \leq C_0 \|g\|_{L^{p'}(\mathbb{R}^d)}^2.
\]
From Fubini’s theorem we further deduce
\[
I_2 = \int_{\mathbb{R}} \int_{\mathbb{R}} (T(t - s)g(s)) \overline{g(t)} \, ds \, dt = \int_{\mathbb{R}} \int_{\mathbb{R}} (T(-s)g(s)(T(-t)g(t))) \, ds \, dt = \|T * g\|_{L^q(\mathbb{R}^d)} \|g\|_{L^{p'}(\mathbb{R}^d)}^2,
\]
where all integrals are C- or \( L^2 \)-valued Riemann integrals. We have thus shown
\[
\|\int_{\mathbb{R}} T(-t)g(t) \, dt\|_2 \leq \sqrt{C_0} \|g\|_{L^{p'}(\mathbb{R}^d)} \tag{11.15}
\]
for \( g \in C_c(\mathbb{R}, L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)) \). Let \( \varphi \in C_c(\mathbb{R}^d) \). Observe that the scalar function \((T(\cdot)\varphi, \overline{g(\cdot)})_{L^q} = (T(\cdot)\varphi)g)_{L^2} \) is measurable. Estimate (11.15) leads to
\[
\left| \int_{\mathbb{R}} (T(t)\varphi, \overline{g(t)})_{L^2} \, dt \right| = \left| \int_{\mathbb{R}} (T(t)\varphi)g(t)_{L^2} \, dt \right| = \left| \int_{\mathbb{R}} (\varphi)(T(-t)g(t))_{L^2} \, dt \right| \leq \sqrt{C_0} \|\varphi\|_2 \|g\|_{L^{p'}(\mathbb{R}^d)}.
\]
Since \( C_c(\mathbb{R}, L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)) \) is dense in \( L^q(\mathbb{R}, L^q(\mathbb{R}^d)) \) by Lemma 11.5, Remark 11.4 yields that \( T(\cdot) \varphi \in L^q(\mathbb{R}, L^q(\mathbb{R}^d)) \) and
\[
\|T(\cdot) \varphi\|_{L^q(\mathbb{R}, L^p)} = \sup_{\|g\|_{L^q} \leq 1} \|\langle T(\cdot) \varphi, g \rangle_{L^q(\mathbb{R}, L^p)}\| \leq \sqrt{C_0} \|\varphi\|_2.
\]
By approximation, we derive (a) for \( \varphi \in L^2(\mathbb{R}^d) \) and \( q \in (2, \infty) \).

3) Let \( k = 0 \), \((q, p) = (\infty, 2)\) and \((q, p)\) be admissible with \( q > 2 \). Then (a) holds since \( T(\cdot) \) is a unitary \( C_0 \)-group on \( L^2(\mathbb{R}^d) \). To prove (b), we set \( f_t = \mathbb{1}_{[0, t]}f \) for \( t > 0 \) and \( f_t = \mathbb{1}_{[t, 0]}f \) for \( t < 0 \). We write \((q, p)\) instead of \((q, p)\).
First let \( f \in C_c(\mathbb{R}, L^2(\mathbb{R}^d) \cap L^{p'}(\mathbb{R}^d)) \). Using that \( T(t) \) is isometric and strongly
continuous on $L^2(\mathbb{R}^d)$ and inequality (11.15), we obtain $T_{*+}f \in C_b(\mathbb{R}, L^2(\mathbb{R}^d))$
and
$$\| T_{*+} f \|_{C_b(\mathbb{R}, L^2)} = \sup_{t \in \mathbb{R}} \| \int_0^t T(t-s) f(s) \, ds \|_2 = \sup_{t \in \mathbb{R}} \| \int_\mathbb{R} T(-s) f(s) \, ds \|_2$$
$$\leq \sup_{t \in \mathbb{R}} \sqrt{C_0} \| f \|_{L^{q'}(\mathbb{R}, L^{p'})} = \sqrt{C_0} \| f \|_{L^{q'}(\mathbb{R}, L^{p'})}. $$

If we approximate the given $f$ in $L^q(\mathbb{R}, L^p(\mathbb{R}^d))$ by $f_n \in C_c(\mathbb{R}, L^2(\mathbb{R}^d) \cap L^p(\mathbb{R}^d))$, then the above estimate shows that $(T_{*+} f_n)_{n}$ converges to a function $u$ in $C_b(\mathbb{R}, L^2(\mathbb{R}^d))$. On the other hand, step 1) implies that for a subsequence the functions $(T_{*+} f_{n_j})(t)$ tend to $(T_{*+} f)(t)$ in $L^p(\mathbb{R}^d)$ for a.e. $t \in \mathbb{R}$. Hence, $T_{*+} f = u$ belongs to $C_b(\mathbb{R}, L^2(\mathbb{R}^d))$ and (b) is true in the present case.

4) Let $k = 1$. By Lemma 11.3 the spatial derivatives $\partial_j (j = 1, \ldots, d)$ commute with $T(t)$ on $H^1(\mathbb{R}^d)$. This fact easily implies that (a) holds with $k = 1$ and that $T(\cdot) \varphi \in C_b(\mathbb{R}, L^1(\mathbb{R}^d))$ if $\varphi \in H^1(\mathbb{R}^d)$.

For (b), take $f \in L^q(\mathbb{R}, W^{1,q}_p(\mathbb{R}^d))$. As in step 1) we approximate $f$ in $L^q(\mathbb{R}, W^{1,q}_p(\mathbb{R}^d))$ by $g_n \in C_c(\mathbb{R}, H^m(\mathbb{R}^d) \cap L^p(\mathbb{R}^d))$, where $m > 1 + \frac{d}{2} - \frac{d}{p}$ so that $H^m(\mathbb{R}^d) \hookrightarrow W^{1,q}_p(\mathbb{R}^d)$ by Corollary D.16. Step 1) shows that the functions $\partial_j T_{*+} g_n = T_{*+} \partial_j g_n$ tend to $T_{*+} \partial_j f$ in $L^q(\mathbb{R}, L^p(\mathbb{R}^d))$ as $n \to \infty$. As a result, $T_{*+} f$ belongs to $L^q(\mathbb{R}, W^{1,q}_p(\mathbb{R}^d))$ and (b) holds for $k = 1$. The case $k \geq 2$ is treated similarly.

Let $J \subseteq \mathbb{R}$ be an interval with $0 \in J$. Part (a) for $J$ then follows from the assertion for $J = \mathbb{R}$. To derive (b), we extend $f \in L^q(J, W^{1,q}_p(\mathbb{R}^d))$ by 0 to $\tilde{f} \in L^q(\mathbb{R}, W^{1,q}_p(\mathbb{R}^d))$. Then $T_{*+} f = T_{*+} \tilde{f}$ on $J$ and so (b) is also true for $J$. □

Part (a) of Theorem 11.6 is wrong for non-admissible $(q, p)$ as we see in the next remark, whereas part (b) is true for some non-admissible exponents, see §2.4 of [Caz03]. Strichartz’ estimates fail for $(2, \infty)$ if $d = 2$, see [MS98].

REMARK 11.7. A scaling argument shows that Theorem 11.6 (a) can only be valid for admissible exponents. In fact, let $\varphi \in H^2(\mathbb{R}^d) \setminus \{0\}$ and $u = T(\cdot) \varphi$. For $\lambda > 0$, we define $\varphi_\lambda(x) = \varphi(\lambda x)$, $x \in \mathbb{R}^d$. Clearly $\varphi_\lambda \in H^2(\mathbb{R}^d)$ and the solution $u_\lambda = T(\cdot) \varphi_\lambda$ of $iu' = -\Delta u$ with initial value $\varphi_\lambda$ is given by $(u_\lambda(t))(x) = u(\lambda^2 t, \lambda x)$. Let $E = L^q(J, L^p(\mathbb{R}^d))$ for some $1 \leq p, q < \infty$. Observe that
$$\| \varphi_\lambda \|_2 = \left( \int_{\mathbb{R}^d} |\varphi(\lambda x)|^2 \, dx \right)^{\frac{1}{2}} = \left( \int_{\mathbb{R}^d} \lambda^{-d} |\varphi(y)|^2 \, dy \right)^{\frac{1}{2}} = \lambda^{-\frac{d}{2}} \| \varphi \|_2,$$
$$\| u_\lambda \|_E = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(\lambda^2 t, \lambda x)|^p \, dx \right)^{\frac{q}{p}} \, dt \right)^{\frac{1}{q}} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |u(s, y)|^p \lambda^{-d} \, ds \right)^{\frac{q}{p}} \lambda^{-2} \, dy \right)^{\frac{1}{q}} = \lambda^{-\frac{d}{q}} \lambda^{-\frac{d}{2}} \| u \|_E.$$  

Suppose that Theorem 11.6 (a) holds for $(q, p)$. Then
$$\lambda^{-\frac{d}{p} - \frac{2}{q}} \| u \|_E = \| u_\lambda \|_E \leq c \| \varphi_\lambda \|_2 = \lambda^{-\frac{d}{2}} c \| \varphi \|_2$$
for a constant $c > 0$ and all $\lambda > 0$. Hence, $\lambda^{\frac{d}{2} - \frac{d}{p} - \frac{2}{q}}$ is uniformly bounded for $\lambda > 0$ and thus $\frac{d}{2} = \frac{d}{p} + \frac{2}{q}$. ◊
Exercises

Exercise 11.1 (Scaling). Let $\varphi \in H^2(\mathbb{R}^d)$ and let $u$ be an $H^2$–solution of (11.1) on $J$ with $u(0) = \varphi$. For $\lambda > 0$ and $\kappa \in \mathbb{R}$ define the function $(u_\lambda(t))(x) = \lambda^\kappa(u(\lambda^2 t))(\lambda x)$, $t \in J$, $x \in \mathbb{R}^d$. For which exponent $\kappa = \kappa(\alpha)$ does the map $u_\lambda$ solve (11.1) for the initial value $\varphi_\lambda$ given by $\varphi_\lambda(x) = \lambda^\kappa \varphi(\lambda x)$?

Fix this exponent $\kappa(\alpha) = \kappa$ in the definition of $u_\lambda$. For which $\alpha > 1$ we then have $\|u_\lambda(t)\|_2 = \|u(\lambda^2 t)\|_2$ or $\|\partial_k u_\lambda(t)\|_2 = \|\partial_k u(\lambda^2 t)\|_2$ for all $t \in J$ and $k \in \{1, \ldots, d\}$?

Exercise 11.2 (Symmetries). Let $u$ be an $H^2$–solution of (11.1) on $J = \mathbb{R}$ (see Definition 10.8). For $h \in \mathbb{R}^d$, $Q \in \mathbb{R}^{d \times d}$ and $\varphi : \mathbb{R}^d \to \mathbb{C}$ we set $(S_h \varphi)(x) = \varphi(x - h)$ and $(R_Q \varphi)(x) = \varphi(Q x)$, $x \in \mathbb{R}^d$. We define

$$w_1(t) = e^{i\theta} S_{x_0} u(t - t_0)$$

for fixed $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^d$ and $\theta \in \mathbb{R},$

$$w_2(t) = u(-t),$$

$$w_3(t) = R_Q u(t)$$

for a fixed orthogonal $Q \in \mathbb{R}^{d \times d},$

$$w_4(t) = e^{i \varphi} e^{-|\varphi|^2 t} S_{2\varphi} u(t)$$

for a fixed $\varphi \in \mathbb{R}^d,$

where $t \in \mathbb{R}$ and $e_{i\varphi}(x) = e^{i \varphi \cdot x}$. Show that the functions $w_j$ ($j = 1, 2, 3, 4$) satisfy (11.1) for the appropriate initial values. Further show that $u(t)$ is spherically symmetric for all $t \in \mathbb{R}$ if $u(0)$ is spherically symmetric.
Bibliography


